Towards an Optimal Outdoor Advertising Placement: When a Budget Constraint Meets Moving Trajectories

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In this article, we propose and study the problem of trajectory-driven influential billboard placement: given a set of billboards U (each with a location and a cost), a database of trajectories $\mathcal{T}$, and a budget L, we find a set of billboards within the budget to influence the largest number of trajectories. One core challenge is to identify and reduce the overlap of the influence from different billboards to the same trajectories, while keeping the budget constraint into consideration. We show that this problem is NP-hard and present an enumeration based algorithm with $(1 - 1/e)$ approximation ratio. However, the enumeration would be very costly when $|U|$ is large. By exploiting the locality property of billboards' influence, we propose a partition-based framework PartSel. PartSel partitions U into a set of small clusters, computes the locally influential billboards for each cluster, and merges them to generate the global solution. Since the local solutions can be obtained much more efficiently than the global one, PartSel would reduce the computation cost greatly; meanwhile it achieves a non-trivial approximation ratio guarantee. Then we propose a LazyProbe method to further prune billboards with low marginal influence, while achieving the same approximation ratio as PartSel. Next, we propose a branch-and-bound method to eliminate unnecessary enumerations in both PartSel and LazyProbe, as well as an aggregated index to speed up the computation of marginal influence. Experiments on real datasets verify the efficiency and effectiveness of our methods.

CCS Concepts: • **Information systems** → *Information systems applications*;

Additional Key Words and Phrases: Outdoor advertising, influence maximization, trajectory

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1 INTRODUCTION

Outdoor advertising (ad) has a $500 billion global market; its revenue has grown by over 23% in the past decade to over $6.4 billion in the US alone [2]. As compared to social, television, and mobile advertising, outdoor advertising delivers a high return on investment, and according to [1] an average of $5.97 is generated in product sales for each dollar spent. Moreover, it literally drives consumers "from the big screen to the small screen" to search, interact, and transact [4]. Billboards are the highest-used medium for outdoor advertising (about 65%), and 80% people notice them when driving [5].

Nevertheless, existing market research only leverages traffic volume to assess the performance of billboards [19]. Such a straight-forward approach often leads to coarse-grained performance estimations and undesirable ad placement plans. To enable more effective placement strategies, we propose a fine-grained approach by leveraging the user/vehicle trajectory data. Enabled by the prevalence of positioning devices, tremendous amounts of trajectories are being generated from vehicle Global Positioning System devices, smart phones, and wearable devices. The massive trajectory data provides new perspective to assess the performance of ad placement strategies.

In this article, we propose a quantitative model to capture the billboard influence over a database of trajectories. Intuitively, if a billboard is close to a trajectory along which a user or vehicle travels, the billboard influences the user to a certain degree. When multiple billboards are close to a trajectory, the marginal influence is reduced to capture the property of diminishing returns. Based on this influence model, we propose and study the the Trajectory-driven Influential Billboard Placement (TIP) problem: given a set of billboards, a database of trajectories and a budget constraint L, it finds a set of billboards within budget L such that the placed ads on the selected billboards influence the largest number of trajectories. To the best of our knowledge, this is the first work to address the TIP problem. The primary goal of this article is to maximize the influence within a budget, which is critical to advertisers because the average unit cost per billboard is not cheap. For example, the average cost of a unit is $14,000 for 4 weeks in New York [3]; the total cost of renting 500 billboards is $7,000,000 per month. Since the cost of a billboard is usually proportional to its influence, if we can improve the influence by 5%, we can save about $10,000 per week for one advertiser. The secondary goal is how to avoid expensive computation while achieving the same competitive influence value, so that prompt analytic on deployment plans can be conducted with different budget allocations.

In particular, there are two fundamental challenges to achieve the above goals. First, a user's trajectory can be influenced by multiple billboards, which incurs the influence overlap among billboards. Figure 1 shows an example for six billboards $(b_1, \ldots, b_6)$ and six trajectories $(t_1, \ldots, t_6)$. Each billboard is associated with a λ-radius circle, which represents its influence range. If any point p in a trajectory t lays in the circle of b, t is influenced by b with a certain probability. Thereby, trajectory t_1 is first influenced by billboard b_1 and then influenced by b_3. If the selected billboards have a large overlap in their influenced trajectories, advertisers may waste the money for repeatedly influencing the audiences who have already seen their ads. Second, the budget constraint L and various costs of different billboards make the optimization problem intricate. To our best knowledge, this is the first work that simultaneously takes three critical real-world

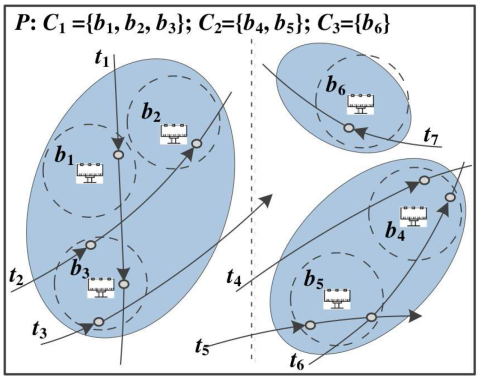


Fig. 1. A motivating example ($w(b_i) = i$).

features into consideration, i.e., budget constraint, non-uniform costs of billboards, and influence overlap of the selected billboards to a certain trajectory (Section 2).

To address these challenges, we first propose a greedy framework EnumSel by employing the enumeration technique [13], which can provide an $(1 - 1/e)$-approximation for TIP. However the algorithm runs in a prohibitively large complexity of $O(|\mathcal{T}| \cdot |U|^5)$, where $|\mathcal{T}|$ and $|U|$ are the number of trajectories and billboards respectively. To avoid such high computational cost, we exploit the locality feature of the billboard influence and propose a partition-based framework. The core idea works as follows: first, it partitions the billboards into a set of clusters with low influence overlap; second, it executes the enumeration algorithm to find local solutions; and third, it uses the dynamic programming approach to construct the global solution based on the location solutions maintained by different clusters. We prove that the partition based method provides a theoretical approximation ratio. Moreover, we devise a lazy probe approach by pro-actively estimating the upper bound of each cluster and combining the results from a cluster only when its upper bound is significant enough to contribute to the global solution. However, the enumeration and the marginal influence computation are still two bottlenecks of the proposed algorithms. To overcome these issues, we try to speed up the above methods by utilizing the distribution of data. The core idea has two folds: (1) we employ a branch-and-bound method to eliminate unnecessary enumerations in PartSel and LazyProbe, which can reduce the runtime greatly if the data contains many low cost-effective billboards; and (2) we build an aggregated index on $\mathcal{T}$ to group similar trajectories together, thus enabling us to compute the marginal influence on trajectory clusters rather than dealing with each individual trajectory in $\mathcal{T}$ one by one.

Beyond billboard selection, our solution is useful in any store site selection problem that needs to consider the influence gain w.r.t. the cost of the store under a budget constraint. The only change is a customization of the influence model catered for specific scenarios, while the influence overlap is always incurred whenever the audiences are moving. For example, in the electric vehicle charging station deployment, each station has an installment fee and a service range, which is similar to the billboard in TIP. Given a budget limit, its goal is to maximize the deployment benefit, which can be measured by the trajectories that can be serviced by the stations deployed. In summary, we make the following contributions.

- We formulate the problem of TIP (Section 2), and present a greedy algorithm with the enumeration technique (EnumSel) as the baseline solution to achieve a $(1 - 1/e)$ approximation for TIP (Section 3). To our best knowledge, this is the first work that simultaneously takes

three critical real-world features into consideration, i.e., budget constraint, unequal costs of billboards, and influence overlap of the selected billboards to a certain trajectory.
- We propose a partition-based framework (PartSel) by exploiting the locality property of the influence of billboards. PartSel significantly reduces the computation cost while achieving a theoretical approximation ratio (Section 4).
- We propose a LazyProbe method to further prune billboards with low benefit/cost ratio, which significantly reduces the practical cost of PartSel while achieving the same approximation ratio (Section 5).
- We propose a pruning method to eliminate the candidates in U that would not be in the answer set, thereby reducing the number of enumerations and speeding up EnumSel, PartSel, and LazyProbe. Moreover, we aggregate the trajectories influenced by the same billboard set together, and introduce an index structure to accelerate the marginal influence computation (Section 6).
- We conduct extensive experiments on real-world trajectory and billboard datasets. The results demonstrate the superiority of the proposed methods over the traditional approaches (Section 7).

2 PRELIMINARY

In this section, we first formulate our problem, and then review the relevant studies and justify their differences to our work.

2.1 Problem Formulation

In a trajectory database $\mathcal{T}$, each (human or vehicle) trajectory t is in the form of a sequence of locations $t = \{p_1, p_2, \ldots, p_{|t|}\}$; a trajectory location p_i is represented by $\{lat, lng\}$, where lat and lng represent the latitude and longitude respectively. A billboard b is in the form of a tuple $\{loc, w\}$, where loc and w denote b's location and leasing cost respectively. Without loss of generality, we assume that a billboard carries either zero or one advertisement at any time.

Definition 2.1. We define that b can influence t, if $\exists p_i \in t$, such that $Distance(p_i, b.loc) \leq \lambda$, where $Distance(p_i, b.loc)$ computes a certain distance between p_i and $b.loc$, and λ is a given threshold.

The choice of distance functions is orthogonal to our solution, and we choose Euclidean distance for illustration purpose.

Influence of a billboard b_i to a trajectory t_j, $pr(b_i, t_j)$. Given a trajectory t_j and a billboard b_i that can influence t_j, $pr(b_i, t_j)$ denotes the influence of b_i to t_j. The influence can be measured in various ways depending on application needs, such as the panel size, the exposure frequency, the travel speed and the travel direction. Note that our solutions of finding the optimal placement is orthogonal to the choice of influence measurements, so long as it can be computed deterministically given a b_i and t_j. By looking into the influence measurement of one of the largest outdoor advertising companies LAMAR [3], we observe that panel size and exposure frequency are used. Moreover, these two can be obtained from the real data, hence we adopt them in our influence model and experiment. (1) For all $b_i \in U$ and $t_j \in \mathcal{T}$, we set $pr(b_i, t_j)$ as a uniform value (between 0 and 1) if b_i can influence t_j. (2) Let $size(b_i)$ be the panel size of b_i. We set $pr(b_i, t_j) = size(b_i)/A$ for t_j influenced by b_i, where A is a given value that is larger than $\max_{b_i \in U} size(b_i)$.

Influence of a billboard set S to a trajectory t_j, $pr(S, t_j)$. It is worth noting that $pr(S, t_j)$ cannot be simply computed as $\sum_{b_i \in S} pr(b_i, t_j)$, because different billboards in S may have overlaps when they influence t_j. Obviously $pr(S, t_j)$ should be the probability that at least one billboard in S can

influence t_j. Thus, we use the following equation to compute the influence of S to t_j:

$$pr(S, t_j) = 1 - \prod_{b_i \in S} (1 - pr(b_i, t_j)) \tag{1}$$

where $(1 - pr(b_i, t_j))$ is the probability that b_i cannot influence t_j.

Influence of a billboard set S to a trajectory set $\mathcal{T}$, $I(S)$. Let $\mathcal{T}_S$ denote the set of trajectories in $\mathcal{T}$ that are influenced by at least one billboard in S. The influence of a billboard set S to a trajectory set $\mathcal{T}$ is computed by summing up $pr(S, t_j)$ for $t_j \in \mathcal{T}_S$:

$$I(S) = \sum_{t_j \in \mathcal{T}_S} pr(S, t_j) \tag{2}$$

Example 2.1. Let $S = \{b_1, b_2, b_3\}$ be a set of billboards chosen from all billboards in Figure 1, and trajectories t_1, t_2, and t_3 that are influenced by at least one billboard in S. Let $pr(b_1, t_1) = 0.1$, $pr(b_3, t_1) = 0.3$, and $pr(b_2, t_1) = 0$ (b_2 does not influence t_1). By Equation (1), we have $pr(S, t_1) = 1 - (1 - pr(b_1, t_1)) \times (1 - pr(b_3, t_1)) = 1 - (1 - 0.1) \times (1 - 0.3) = 0.37$. Similarly, we have $pr(S, t_2) = 0.44$ and $pr(S, t_3) = 0.3$. Finally, the total influence of S is equal to $pr(S, t_1) + pr(S, t_2) + pr(S, t_3) = 1.11$.

Definition 2.2 (Trajectory-driven Influential Billboard Placement). Given a trajectory database $\mathcal{T}$, a set of billboards U to place ads and a cost budget L from a client, our goal is to select a subset of billboards $S \subset U$, which maximizes the expected number of influenced trajectories such that the total cost of billboards in S does not exceed budget L.

Theorem 2.1. *The TIP problem is NP-hard.*

Proof. We prove it by reducing the Set Cover problem to the TIP problem. In the Set Cover problem, given a collection of subsets $S_1, \ldots, S_m$ of a universe of elements U', we wish to know whether there exist k of the subsets whose union is equal to U'. We map each element in U' in the Set Cover problem to each trajectory in $\mathcal{T}$. We also map each subset S_i to the set of trajectories influenced by a billboard b_i. Consequently, if all the trajectories in U' are influenced by S, the influence of S is $|U'|$. Subsequently, the cost of each billboard is set to 1 and budget L in TIP is set to k (selecting only k billboards). The Set Cover problem is equivalent to deciding if there is a k-billboard set with the maximum influence U' in the TIP problem. As the set cover problem is NP-complete, the decision problem of TIP is NP-complete, and the optimization problem is NP-hard. □

2.2 Related Work

Maximized Bichromatic Reverse k Nearest Neighbor (MaxBRkNN). The MaxBRkNN queries [9, 20, 28, 29] aim to find the optimal location to establish a new store such that it is a kNN of the maximum number of users based on the spatial distance between the store and users' locations. Different spatial properties are exploited to develop efficient algorithms, such as space partitioning [29], intersecting geometric shapes [28], and sweep-line techniques [20]. Recently, the MaxRKNN query [26] is proposed to find the optimal bus route in term of maximum bus capacity by considering the audiences' source-destination trajectory data. Regarding the usage of trajectory data, most recent work only focus on top-k search over trajectory data [25, 27].

Our TIP problem is different from MaxBRkNN in two aspects. (1) MaxBRkNN assumes that each user is associated with a fixed (check-in) location. In reality, the audience can meet more than one billboard while moving along a trajectory, which is captured by the TIP model. Thus it is challenging to identify such influence overlap when those billboards belong to the same placement strategy.

(2) Billboards at different locations may have different costs, making this budget-constrained optimization problem more intricate. However, MaxBRkNN assumes that the costs of candidate store locations are uniform.

Influence Maximization and its variations. The original Influence Maximization (IM) problem aims to find a size-k subset of all nodes in a social network that could maximize the spread of influence [12]. Independent Cascade (IC) model and Linear Threshold (LT) model are two common models to capture the influence spread. Under both models, this problem has been proven to be NP-hard, and a simple greedy algorithm guarantees the best possible approximation ratio of $(1 - 1/e)$ in polynomial time. Then the key challenge lies in how to calculate the influence of sets efficiently, and a plethora of algorithms [6–8, 14, 23] have been proposed to achieve speedups. Some new models are also introduced to solve IM under complex scenarios. IM problems for propagating different viral products are studied in [16, 17]. Recently, the IM problem is extended to location-aware IM (LIM) problems by considering different spatial contexts [11, 15, 19]. Li et al. [15] find the seed users in a location-aware social network such that the seeds have the highest influence upon a group of audiences in a specified region. Guo et al. [11] select top-k influential trajectories based on users' check-in locations. See a recent survey [18] for more details.

Our TIP differs from the IM problems as follows. (1) The cardinality of the optimal set in IM problems is often pre-determined because the cost of each candidate is equal to each other (when the cost is 1, the cardinality is k), thus a theoretically guaranteed solution can be directly obtained by a naive greedy algorithm. However, in our problem, the costs of billboards at different locations differ from one to another, so the theoretical guarantee of the naive greedy algorithm is poor [13]. (2) Since IM problems adopt a different influence model to ours, they mainly focus on how to efficiently and effectively estimate the influence propagation, while TIP focuses on how to optimize the profit of k-combination by leveraging the geographical properties of billboards and trajectories.

Maximum k-coverage problem. Given a universe of elements U and a collection S of subsets from U, the Maximum k-coverage problem (MC) aims to select at most k sets from S to maximize the number of elements covered. This problem has been shown to be NP-hard, and Feige [10] has proven that the greedy heuristic is the most effective polynomial solution and can provide $(1 - 1/e)$ approximation to the optimal solution. The budgeted maximum coverage (BMC) problem [13] further considers a cost for each subset and tries to maximize the coverage with a budget constraint. Khuller et al. [13] show that the naive greedy algorithm no longer produces solutions with an approximation guarantee for BMC. To overcome this issue, they devise a variant of the greedy-based algorithm for BMC, which provides solutions with a $(1 - 1/e)$-approximation. However, by a rigorous complexity analysis in Section 3.1.2, we find that this algorithm needs to take $O(|\mathcal{T}| \cdot |U|^5)$ time to solve our TIP problem, which does not scale well in practice (see Section 7).

3 OUR FRAMEWORK

We first discuss two baselines that are extended from the algorithms for the general BMC problem. In particular, we first present a basic greedy method (Algorithm 1). It is worth noting that the basic greedy method is proved by Khuller et al. [13] to achieve $(1 - 1/\sqrt{e})$-approximation; however, we find it is not correct and we prove it to be $\frac{1}{2}(1 - 1/e)$. As the approximation ratio of this algorithm is low, we then propose an enumeration algorithm with $(1 - 1/e)$-approximation (Algorithm 2). However, the enumeration algorithm incurs a high computation cost as it has to enumerate a large number of feasible candidate combinations, which is impractical when $|U|$ and $|\mathcal{T}|$ are large. This motivates us to exploit the spatial property between billboards and trajectories to propose our own framework to dramatically reduce the computation cost, where an overview is shown in Section 3.2. Important notations used in our framework are presented in Table 1.

3.1 Baselines

3.1.1 A Basic Greedy Method. A straightforward approach is to select the billboard b which maximizes the unit marginal influence, i.e., $\frac{\Delta(b|S)}{w(\{b\})}$, to a candidate solution set S, until the budget is exhausted, where $\Delta(b|S)$ denotes the marginal influence of b to S, i.e., $I(S \cup \{b\}) - I(S)$. Lines 1.3–1.8 of Algorithm 1 present how it works. However, such a greedy heuristic cannot achieve a guaranteed approximation ratio. For example, given two billboards b_1 with influence 1 and b_2 with influence x. Let $w(b_1) = 1$, $w(b_2) = x + 1$ and $L = x + 1$. The optimal solution is b_2 which has influence x, while the solution picked by the greedy heuristic contains the set b_1 and the influence is 1. The approximation factor for this instance is x. As x can be arbitrarily large, this greedy method is unbounded.

To overcome this issue, we modify the above method by considering the best single billboard solution as an alternative to the output of the naive greedy heuristic. In particular, we add lines 1.9–1.13 in Algorithm 1 to consider such best single billboard solution. As a result, a complete Algorithm 1 forms our basic greedy method (GreedySel) to solve the TIP problem.

ALGORITHM 1: GreedySel (U, L, S)

```
1.1  Input: A billboard set U, a budget L and a set S (S = φ by default)
1.2  Output: A billboard set S ⊆ U such that w(S) ≤ L
1.3  repeat
1.4      Select b ∈ U \ S that maximizes Δ(b|S) / w({b})
1.5      if w(S) + w(b) ≤ L then
1.6          S ← S ∪ {b}
1.7      U ← U \ {b}
1.8  until U = φ;
1.9  H ← argmax{I({b})| b ∈ U, and w({b}) ≤ L}
1.10 if I(H) > I(S) then
1.11     return H
1.12 else
1.13     return S
```

Time Complexity of GreedySel. In each iteration, Algorithm 1 needs to scan all the billboards in $(U \setminus S)$ and compute their (unit) marginal influence to the chosen set. Each marginal influence computation needs to traverse $\mathcal{T}$ once in the worst case. Thus, adding one billboard into S takes $O(|\mathcal{T}| \cdot |U|)$ time. Moreover, when L is sufficiently large, this process would repeat $|U|$ times at the worst case. Therefore, the time complexity of Algorithm 1 is $O(|\mathcal{T}| \cdot |U|^2)$.

It is worth noting that the authors in [13] claim that GreedySel achieves an approximation factor of $(1 - 1/\sqrt{e})$ for the BMC problem. However, we find that this claim is problematic and the bound of GreedySel should be $\frac{1}{2}(1 - 1/e)$, as presented in Theorem 3.1.

Theorem 3.1. *GreedySel achieves an approximation factor of $\frac{1}{2}(1 - 1/e)$ for the TIP problem.*

Discussion on the problematic approximation ratio of $(1 - \frac{1}{\sqrt{e}})$ originally presented in [13]). Note that Theorem 3.1 is essentially the Theorem 3 introduced in [13] because both try to find the approximation ratio of the same cost-effective greedy method for a BMC problem. We first present a proof of Theorem 3.1 which shows that the GreedySel achieves $\frac{1}{2}(1 - 1/e)$-approximation, then we justify why the approximation ratio of $(1 - \frac{1}{\sqrt{e}})$ originally presented in [13]) is problematic.

Proof (Theorem 3.1). Let OPT denote the optimal solution and $\mathbb{M}_{k^*+1}$ be the marginal influence of adding b_{k^*+1} (be consistent to the definition in Lemma 5.3). When applying Lemma 5.2 to the (k^*+1)-th iteration, we get:

$$\begin{aligned} I(S_{k^*+1}) &= I(S_{k^*} \cup b_{k^*+1}) = I(S_{k^*}) + \mathbb{M}_{k^*+1} \\ &\geq \left[1 - \prod_{j=1}^{k^*+1}\left(1 - \frac{w(b_j)}{L}\right)\right] \cdot I(OPT) \\ &\geq \left(1 - \left(1 - \frac{1}{k^*+1}\right)^{k^*+1}\right) \cdot I(OPT) \\ &\geq \left(1 - \frac{1}{e}\right) \cdot I(OPT) \end{aligned}$$

Note that the second inequality follows from the fact that adding b_{k^*+1} to S violates the budget constraint L, i.e., $w(S_{k^*+1}) = w(S_{k^*}) + w(b_{k^*+1}) \geq L$.

Intuitively, $\mathbb{M}_{k^*+1}$ is at most the maximum influence of the elements covered by a single billboard, i.e., H is found by GreedySel in the first step (line 1.3). Moreover, as $S_{k^*} \subseteq S$ (S: the solution of GreedySel), we have:

$$I(S) + I(H) \geq I(S_{k^*+1}) \geq (1 - 1/e)I(OPT) \tag{3}$$

From the above inequality we have that, among $I(S)$ and $I(H)$, at least one of them is no less than $\frac{1}{2}(1 - 1/e)I(OPT)$. Thus it shows that GreedySel achieves an approximation ratio of at least $\frac{1}{2}(1 - 1/e)$. □

In the original proof of Theorem 3 in [13], the authors have tried to prove that GreedySel is $(1 - 1/\sqrt{e})$-approximate for the following three cases, respectively.

Case 1: the influence of the most influential billboard in U is greater than $\frac{1}{2}I(OPT)$.

Case 2: no billboard in U has an influence greater than $\frac{1}{2}I(OPT)$ and $w(S) \leq \frac{1}{2}L$.

Case 3: no billboard in U has an influence greater than $\frac{1}{2}I(OPT)$ and $w(S) \geq \frac{1}{2}L$.

The authors also proved that the approximate factor in Theorem 3.1 can be further tightened to $\frac{1}{2}$ for Case 1 and Case 2, which are right. However, there is a problem in the proof for Case 3. Intuitively, if we can prove that GreedySel is $(1 - 1/\sqrt{e})$-approximate in Case 3, then by the union bound GreedySel can achieve an approximation factor of $(1 - 1/\sqrt{e})$.

Let $w(S_{k^*})$ be equal to γL and $\gamma \in (0, 1)$. By applying Lemma 5.2 to the k^*-th iteration, we get:

$$\begin{aligned} I(S) \geq I(S_{k^*}) &\geq \left[1 - \prod_{j=1}^{k^*}\left(1 - \frac{w(\{b_j\})}{L}\right)\right] \cdot I(OPT) \\ &\geq \left(1 - \left(1 - \frac{1}{\frac{1}{\gamma}k^*}\right)^{k^*}\right) \cdot I(OPT) \\ &\geq \left(1 - \frac{1}{e^{\gamma}}\right) \cdot I(OPT) \end{aligned}$$

Note that $w(S) \geq \frac{1}{2}L$ cannot guarantee $\gamma \geq 1/2$ because $S_{k^*} \subseteq S$. Consequently, the inequality cannot guarantee $I(S) \geq (1 - \frac{1}{\sqrt{e}}) \cdot I(OPT)$. However, it is concluded in [13] that GreedySel achieves an approximation factor of $(1 - \frac{1}{\sqrt{e}})$ under the assumption of $\gamma \geq 1/2$. Therefore, the proof in [13] is problematic.

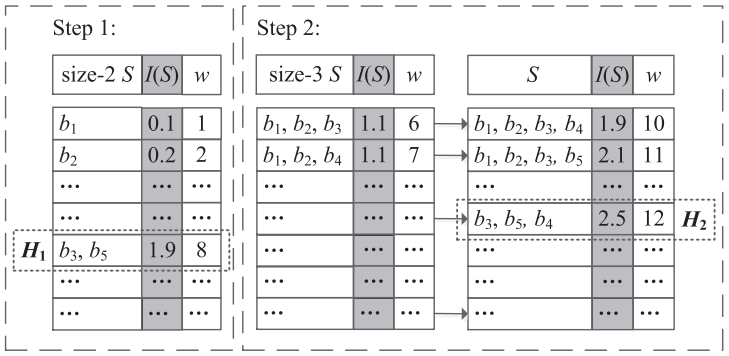


Fig. 2. A running example of Algorithm 2.

3.1.2 Enumeration Greedy Algorithm. Since GreedySel is only $\frac{1}{2}(1-1/e)$-approximation, we would like to further boost the influence value, even at the expense of longer processing time as compared to GreedySel. Note that it is critical to maximize the influence as it can save real money, while keeping acceptable efficiency. Thus we utilize the enumeration-based solution proposed in [13] to obtain $(1-1/e)$-approximation.

EnumSel runs in two phases. In the first phase (line 2.4), it enumerates all feasible billboard sets whose cardinality is no larger than a constant τ, and adds the one with the largest influence to H_1. In the second phase (lines 2.5–2.9), it enumerates each feasible set of size-$(\tau+1)$ whose total cost does not exceed budget L. Then for each set S, it invokes NaiveGreedy to greedily select new billboards (if any) that can bring marginal influence, and chooses the one that maximizes the influence under the remaining budget $L - w(S)$ and assigns it to H_2. Last, if the best influence of all size-$(\tau+1)$ billboard sets is still smaller than that of its size-τ counterpart (i.e., $I(H_1) > I(H_2)$), H_1 is returned; otherwise, H_2 is returned.

Example 3.2. Figure 2 illustrates an instance of Algorithm 2 on Figure 1's scenario. We assume $\tau = 2$ and $L = 12$, and the cost of a billboard is its id number (e.g., $w(b_1) = 1$). For $pr(b_i, t_j)$, we set the influence value of b_i to any trajectory t_j as $i/10$ if t_j can pass b_i (e.g., $pr(b_1, t_1) = 0.1$ and $pr(b_2, t_2) = 0.2$). In the first step, Algorithm 2 enumerates all feasible sets of size less than 3, among which the billboard set $\{b_3, b_5\}$ has the largest influence ($I(H_1) = pr(b_3, t_1) + pr(b_3, t_2) + pr(b_3, t_3) + pr(b_5, t_5) + pr(b_5, t_6) = 1.9$). In the second step, it starts from the feasible size-3 sets and expands

ALGORITHM 2: EnumSel (U, L)

```
2.1  Input: A billboard set U, budget L
2.2  Output: A billboard set S ⊆ U with the cost constraint w(S) ≤ L
2.3  Let τ be a constant /* τ = 2 to achieve the lowest time complexity */
2.4  H1 ← argmax{I(S')| S' ⊆ U, |S'| ≤ τ, and w(S') ≤ L}
2.5  H2 ← ϕ
2.6  for all S ⊆ U, such that |S| = τ + 1 and w(S) ≤ L do
2.7  |   S ← GreedySel(U \ S, L − w(S), S)
2.8  |   if I(S) > I(H2) then
2.9  |   |   H2 ← S
2.10 if I(H1) > I(H2) then
2.11 |   return H1
2.12 else
2.13 |   return H2
```

greedily until the budget constraint is violated. The right part of Figure 2 shows the eventual billboard set S whose total cost does not violate the budget constraint L (line 2.7 of Algorithm 2). Here $w(S) = L = 12$, and assigns it to H_2 (line 2.9), so $H_2 = \{b_3, b_4, b_5\}$ and its influence value $I(H_2) = 2.5$ which is the largest influence. Since $I(H_1) < I(H_2)$, Algorithm 2 returns $\{b_3, b_4, b_5\}$ as the final result.

Time Complexity of EnumSel. At the first phase, Algorithm 2 needs to scan all feasible sets with cardinality τ and the number of such sets is $O(|U|^\tau)$. For each such candidate set, we need to scan $\mathcal{T}$ to compute its influence, thus the first phase takes $O(|\mathcal{T}| \cdot |U|^\tau)$ time. At the second phase, there are $O(|U|^{\tau+1})$ sets of cardinality $\tau + 1$, and Algorithm 2 invokes Algorithm 1 for each set. In the worse case, the cost of any size-$(\tau + 1)$ sets should be much smaller than L and thus these sets would not affect the complexity of GreedySel in line 2.6. Therefore, the second phase takes $O(|\mathcal{T}| \cdot |U|^2 \cdot |U|^{\tau+1})$ time. In total, Algorithm 2 takes $O(|\mathcal{T}| \cdot |U|^\tau + |\mathcal{T}| \cdot |U|^{\tau+3}) = O(|\mathcal{T}| \cdot |U|^{\tau+3})$.

Selection of τ. It has been proved in [13] that Algorithm 2 can achieve an approximation factor of $(1 - 1/e)$ when $\tau \geq 2$. Note that (1) the approximation ratio $(1 - 1/e)$ cannot be improved by a polynomial algorithm [13] and (2) a larger τ leads to larger overhead, thus we set $\tau = 2$. So Algorithm 2 can achieve the $(1 - 1/e)$-approximation ratio with a complexity of $O(|\mathcal{T}| \cdot |U|^5)$.

3.2 A Partition-based Framework

Although EnumSel provides a solution with an approximation ratio of $(1 - 1/e)$, it involves high computation cost, because it needs to enumerate all size-τ and size-$(\tau + 1)$ billboard sets and compute their influence to the trajectories, which is impractical when $|U|$ and $|\mathcal{T}|$ are large. To address this problem, we propose a partition-based framework.

Partition-based Framework. Our problem has a distance requirement that if a billboard influences a trajectory, the trajectory must have a point close to the billboard (distance within λ). All of existing techniques neglect this important feature, which can be utilized to enhance the performance. After deeply investigating the problem, we observe that most trajectories span over a small area in the real world. For instance, around 85% taxi trajectories in New York do not exceed 5 kilometers (see Section 7). It implies that billboards in different areas should have small overlaps in their influenced trajectories, e.g., the number of trajectories simultaneously influenced by two billboards located in Manhattan and Queens is small. Thereby, we exploit such locality features to propose a partition based method called PartSel. Intuitively, we partition U into a set of small clusters, compute the locally influential billboards for each cluster, and merge the local billboards to generate the globally influential billboards of U. Since the local cluster has much smaller number of billboards, this method reduces the computation greatly while keeping competitive influence quality.

Partition. We first partition the billboards to m clusters $C_1, C_2, \ldots, C_m$, where different clusters have no (or little) influence overlap to the same trajectories. Given a budget l_i for cluster l_i, by calling $EnumSel(C_i, l_i)$, we select the locally influential billboard set $S[i][l_i]$ from cluster C_i within budget l_i, where $S[i][l_i]$ has the maximum influence $\xi[i][l_i]$. Next we want to assign a budget to each cluster C_i and take the union of $S[i][l_i]$ as the globally influential billboard set, where $l_1 + l_2 + \cdots + l_m \leq L$. Obviously, we want to allocate the budgets to different clusters to maximize

$$\sum_{i=1}^{m} \xi[i][l_i] \qquad (4)$$
$$s.t.\ l_1 + l_2 + \cdots + l_m \leq L$$

Table 1. Notations for Problem Formulation and Solutions

Symbol	Description
t ($\mathcal{T}$)	A trajectory (database)
U	A set of billboards that a user wants to advertise
L	the total budget of a user
$I(S)$	The influence of a selected billboard set S
P	A billboard partition
ϑ_{ij}	The overlap ratio between clusters
$\Delta(b\|S)$	The marginal influence of b to S
θ	The threshold for a θ-partition
$\mathbb{I}$	The DP influence matrix: $\mathbb{I}[i][l]$ is the maximum influence of the billboards selected from the first i clusters within budget l ($i \leq m$ and $l \leq L$)
ξ	The local influence matrix: $\xi[i][l]$ is the influence returned by $EnumSel(C_i, l)$, i.e., the maximum influence of billboards selected from cluster C_i within budget l

There are two main challenges in this partition-based method. (1) How to allocate the budgets to each cluster to maximize the overall influence? We propose a dynamic programming algorithm to address this challenge (see Section 4). (2) How to partition the billboards to reduce the influence overlap among clusters? We propose a partition strategy to reduce the influence overlap and devise an effective algorithm to generate the clusters (see Section 4).

Lazy Probe. Although the partition-based method significantly reduces the complexities over the enumeration approach, its dynamic programming process has to repeatably invoke EnumSel to probe the partial solution for every cluster in the partition. It is still expensive to compute the local influence by calling $EnumSel(C_i, l_i)$ many times. We find that it is not necessary to compute the real influence value for those clusters which have low influence to affect the final result, thus reducing the number of calls to *EnumSel*. The basic idea is that we estimate an upper bound $\xi^{\uparrow}[i][l_i]$ of the local solution for a given cluster C_i and a budget l_i; and we do not need to compute the real influence $\xi[i][l_i]$, if we find that using this cluster cannot improve the influence value. This method significantly reduces the practical cost of PartSel while achieving the same approximation ratio. There are two challenges in the lazy probe method. (1) How to utilize the bounds to reduce the computational cost (i.e., avoid calling $EnumSel(C_i, l_i)$)? We propose a lazy probe technique (see Section 5). (2) How to estimate the upper bounds while keeping the same approximation ratio as PartSel? We devise an incremental algorithm to estimate the bounds (see Section 5).

4 PARTITION-BASED METHOD

This section proposes a partition-based method which contains three steps to reduce the computation cost:

a. Partition U into a set of clusters according to their influence overlap.
b. Find local influential billboards with regard to each cluster by calling EnumSel.
c. Aggregate these local influential billboards from clusters to obtain the global solution for TIP.

For convenience sake, this section first presents how to select the billboards based on a given partition scheme, and then discuss how to find a good partition that can provide a high performance and a theoretical approximation ratio for our partition-based method.

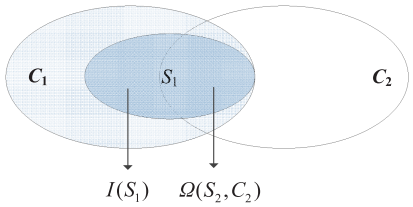


Fig. 3. The relationship of S_1, C_2, and $\Omega(S_1, C_2)$.

4.1 Partition-based Selection Method

Definition 4.1 (Partition). A partition of U is a set of clusters $\{C_1, \ldots, C_m\}$, such that $U = C_1 \cup C_2 \cup \cdots \cup C_m$, and $\forall i \neq j, C_i \bigcap C_j = \phi$. Without loss of generality, we assume that the clusters are sorted by their size, and C_m is the largest cluster.

We follow a divide and conquer framework to combine partial solutions from the clusters. Let S^* denote the billboard set returned by $EnumSel(U, L)$, $S[i][l]$ denote the billboard set returned by $EnumSel(C_i, l)$, where $l < L$ is a budget for cluster C_i, as shown in Figure 3. Let $\xi[i][l]$ be the influence value of the billboard set $S[i][l]$, i.e., $\xi[i][l] = I(S[i][l])$. If $S[i][l]$ for $1 \leq i \leq m$ have no overlap, we can assign a budget l for each cluster and maximize the total influence based on Equation (4).

We note that the costs for billboards are integers in reality, e.g., the costs from a leading outdoor advertising company are all multiples of 100 [3]. Thereby it allows us to design an efficient dynamic programming method to solve Equation (4). The pseudo code is presented in Algorithm 3. It considers the clusters in P one by one. Let $\mathbb{I}[i][l]$ denote the maximum influence value that can be attained with a budget not exceeding l using up to the first i clusters ($i \leq m$ and $l \leq L$). Clearly, $\mathbb{I}[m][L]$ is the solution for Equation (4) since the union of the first m clusters is U. To obtain $\mathbb{I}[m][L]$, Algorithm 3 first initializes the matrices $\mathbb{I}$ and ξ (line 3.3), and then constructs the global solution (line 3.7 to 3.17) with the following recursion:

$$\begin{aligned} &\mathbb{I}[0][l] = 0 \\ &\mathbb{I}[i][l] = \max_{0 \leq q \leq l} (\mathbb{I}[i-1][l-q] + \xi[i][q]) \end{aligned} \tag{5}$$

ALGORITHM 3: PartSel (P, L)

3.1 **Input:** A θ-partition P of U, a budget L
3.2 **Output:** A billboard set S
3.3 Initialize matrices $\mathbb{I}$ and ξ
3.4 $m \leftarrow |P|$
3.5 **for** $i \leftarrow 1$ *to* m **do**
3.6 **for** $l \leftarrow 1$ *to* L **do**
/* C_i is the ith cluster in P */
3.7 Invoke **EnumSel**(C_i, l) to compute $\xi[i][l]$
3.8 $q = \arg\max_{0 \leq q \leq l}(\mathbb{I}[i-1][l-q] + \xi[i][q])$
3.9 $\mathbb{I}[i][l] \leftarrow \mathbb{I}[i-1][l-q] + \xi[i][q]$
3.10 $S \leftarrow$ the corresponding selected set of $\mathbb{I}[m][L]$
3.11 **return** S

Table 2. An Example of Algorithm 3. Each cell in ξ_s and $\mathbb{I}_s$ record the selected billboards corresponding to the maximum influence value recorded in each cell of ξ and $\mathbb{I}$.

(a) ξ_s

	1	2	3
-	-	-	-
C_1	{1}	{1, 3}	{1, 3, 2}
C_2	{6}	{4, 5}	{4, 5, 6}
C_3	{7}	{7, 8}	{7, 8, 9}

(b) ξ

	1	2	3
-	0	0	0
C_1	10	18	25
C_2	8	16	21
C_3	5	9	13

(c) $\mathbb{I}_s$

	1	2	3
$i = 0$	-	-	-
$i = 1$	{1}	{1, 3}	{1, 3, 2}
$i \leq 2$	{1}	{1, 3}	{1, 3, 6}
$i \leq 3$	{1}	{1, 3}	**{1, 3, 6}**

(d) $\mathbb{I}$

	1	2	3
$i = 0$	0	0	0
$i = 1$	10	18	25
$i \leq 2$	10	18	26
$i \leq 3$	10	18	**26**

Since the computation at the ith iteration only relies on the $(i-1)$th row of each matrix, we can use two $2 \times n$ matrices to replace $\mathbb{I}$ and ξ for saving space.

Example 4.1. Given a partition of U as $P = \{C_1, C_2, C_3\}$, where $C_1 = \{1, 2, 3\}$, $C_2 = \{4, 5, 6\}$ and $C_3 = \{7, 8, 9\}$. For simplicity, we assume the cost of each billboard in U is 1. For sake of illustration, we define two more notations: let $\xi_s[i][l]$ and $\mathbb{I}_s[i][l]$ denote the sets of selected billboards corresponding to the influence value $\xi[i][l]$ and $\mathbb{I}[i][l]$ respectively. As a result we have four matrices as shown in Table 2. Now we want to find an influential billboard set within $L = 3$ by Algorithm 3. Initially, $\mathbb{I}[0][l] = 0$, $0 < l \leq 3$. Clearly, for $l = 1, 2 \ldots L$, $\mathbb{I}[1][l]$ is same as $\xi[1][l]$ and $\mathbb{I}_s[1]$ is same as $\xi_s[1]$, as only one cluster is considered. When two clusters are considered: $\mathbb{I}[2][1] = max\{\mathbb{I}[1][1], \mathbb{I}[1][0] + \xi[2, 1]\} = 10$ and $\mathbb{I}_s[1][1] = \{\xi_s[1][1]\} = \{1\}$; $\mathbb{I}[2][2] = max\{\mathbb{I}[1][2], \mathbb{I}[1][0] + \xi[2, 2], \mathbb{I}[1][1] + \xi[2, 1]\} = 18$ and $\mathbb{I}_s[2][2] = \{\xi_s[2][2]\} = \{1, 3\}$. $\mathbb{I}[2][3] = \mathbb{I}[1][2] + \xi[2, 1] = 26$ and $\mathbb{I}_s[2][3] = \{\mathbb{I}_s[1][2] \cup \xi_s[2][1]\} = \{1, 3, 6\}$. This process is repeated until all the elements in $\mathbb{I}$ and $\mathbb{I}_s$ are obtained. Finally, Algorithm 3 returns $\mathbb{I}_s[3][3] = \{1, 3, 6\}$ as a solution, and its influence is $\mathbb{I}[3][3] = 26$.

Time Complexity Analysis. Let $|C_i|$ be the cardinality of C_i. To obtain $\mathbb{I}[i][l]$, Algorithm 3 needs to invoke $EnumSel(C_i, l)$ to compute $\xi[i][l]$ and maximize $\mathbb{I}[i-1][l-q] + \xi[i][q]$. When $\tau = 2$, $EnumSel(C_i, l)$ takes $O(|\mathcal{T}| \cdot |C_i|^5)$ and there are mL elements in $\mathbb{I}$. Therefore, the total time cost of Algorithm 3 is $\sum_{i=1}^{m} |C_i|^5$ which is bounded by $O(mL \cdot |\mathcal{T}| \cdot |C_m|^5)$. It is more efficient than Algorithm 2 ($O(|\mathcal{T}| \cdot |U|^5)$), since $|C_m|$ is often significantly smaller than $|U|$ and L is a constant. As shown in our experiment, PartSel is faster than EnumSel by two orders of magnitude when $|U|$ is 2000.

4.2 θ-partition

A naive partition scheme will lead to poor quality due to large influence overlaps between clusters. In order to reduce the influence overlap between the clusters, we introduce the concept of *Overlap Ratio*. The basic idea is to control the maximum overlap ratio between any subset of a cluster and all the rest clusters.

Definition 4.2 (Overlap Ratio). For two clusters C_i and C_j, the ratio of the overlap between C_i and C_j relative to C_i, denoted by ϑ_{ij}, is defined as:

$$\vartheta_{ij} = \arg\max_{\forall S_i \subseteq C_i} \{\Omega(S_i|C_j)/I(S_i)\} \tag{6}$$

where S_i is a subset of C_i, and $\Omega(S_i|C_j)$ is the overlap between S_i to C_j, i.e., $I(S_i) + I(C_j) - I(S_i \cup C_j)$. The relationship of S_i, C_j, and $\Omega(S_i|C_j)$ is illustrated in Figure 3.

Intuitively, the smaller ϑ_{ij} is, the lower influence overlap that C_i and C_j have.

Given the overlap ratio, we present the concept of θ-partition to trade-off between the cluster size and the overlap of clusters, where θ is a user-defined parameter to control the granularity of the partitions.

Definition 4.3 (θ-partition). Given a threshold θ ($0 \leq \theta \leq 1$), we say a partition $P = \{C_1, .., C_m\}$ is a θ-partition, if $\forall i, j \in [1, m]$ the overlap ratio ϑ_{ij} between any pair of clusters $\{C_i, C_j\}$ is less than θ.

Lemma 4.1. *Let P be a θ-partition of U. Given any set $S \subseteq U$, and the billboards in S belong to k different clusters of P in total. When $k \leq (1/\theta + 1)$, we have $I(S) \geq 1/2 \sum_{S_i \in S} I(S_i)$, where $S_i = S \cap C_i$.*

Proof. To facilitate our proof, we assume $S = \{S_1, S_2 \ldots S_k\}$ and $I(S_1) \geq, \ldots, \geq I(S_k)$. Let $\overline{I(S)}$ denote the average influence among all $S_i \in S$, i.e., $\overline{I(S)} = \frac{1}{k} \sum_{j=1}^{k} I(S_j)$. According to Definition 4.3, we observe that $I(S_i \cup S_j) \geq I(S_i) + (1 - \theta)I(S_j)$, as each subset of S_j has at most θ percent of influence overlapping with the elements of S_i, or vice versa. Then for all subsets of S, we have:

$$
\begin{aligned}
I(S) &\geq I(S_1) + (1-\theta)I(S_2) + (1-2\theta)I(S_3) \ldots + [1-(k-1)\theta]I(S_k) \\
&= \sum_{i=1}^{k} I(S_i) - \theta[I(S_2) + 2I(S_3) + \cdots + (k-1)I(S_k)] \\
&= \sum_{i=1}^{k} I(S_i) - \theta \left[\sum\nolimits_{i=2}^{k} I(S_i) + \sum_{i=3}^{k} I(S_i) + \cdots + I(S_k) \right] \\
&\geq \sum_{i=1}^{k} I(S_i) - \theta[\overline{I(S)} + 2\overline{I(S)} + \cdots + (k-1)\overline{I(S)}] \\
&= \sum_{i=1}^{k} I(S_i) - \theta \frac{k(k-1)}{2} \overline{I(S)}
\end{aligned}
$$

The second inequality above follows from the fact that $\overline{I(S)} \geq \frac{1}{k-1} \sum_{i=j}^{k} I(S_i)$ for $j = 2, 3 \ldots k$, because we have assumed $I(S_1) \geq, \ldots, \geq I(S_k)$. As $k \leq \frac{1}{\theta} + 1$, we have $\theta \frac{k(k-1)}{2} \overline{I(S)} \leq \frac{k}{2} \overline{I(S)}$ and

$$
I(S) \geq \sum_{i=1}^{k} I(S_i) - \frac{k}{2} \overline{I(S)} = 1/2 \sum\nolimits_{S_i \in S} I(S_i) \qquad \square
$$

Based on Lemma 4.1, we proceed to derive the approximation ratio of Algorithm 3 in Theorem 4.2.

Theorem 4.2. *Given a θ-partition $P = \{C_1, \ldots, C_m\}$, Algorithm 3 obtains a $\frac{1}{2}^{\lceil \log_{(1+1/\theta)} m \rceil}(1 - 1/e)$-approximation to the TIP problem.*

Proof. Let S^* and $S = S_1 \cup S_2 \cup \cdots \cup S_k$ S be the solution returned by Algorithm 2 and Algorithm 3 respectively, where $S_i = S \cap C_i$ and $i \leq k \leq m$, i.e., $S = S_1 \cup S_2 \cup \cdots \cup S_k$.

When $\theta = 0$, we have $I(S) = \sum_{i=1}^{k} I(S_i)$. As $\sum_{i=1}^{k} I(S_i)$ is the maximum value of Equation (4), thus $I(S) = \sum_{i=1}^{k} I(S_i) \geq I(S^*)$. Moreover, $I(S^*) \geq (1 - 1/e)I(OPT)$ since S^* is returned by Algorithm 2, thus $I(S) \geq (1 - 1/e)I(OPT)$ and the theorem holds.

When $\theta > 0$, we have $I(S) \leq \sum_{i=1}^{k} I(S_i)$. In this case, let us consider an iterative process. At iteration 0, we denote S^0 as a set of billboard clusters in which cluster S_j^0 corresponds S_j. In each iteration h, we arbitrarily partition the clusters in S^{h-1} and merge each partition to form new disjoint clusters for S^h. Each cluster S_j^h in S^h contains at most $(1 + 1/\theta)$ clusters from S^{h-1}.

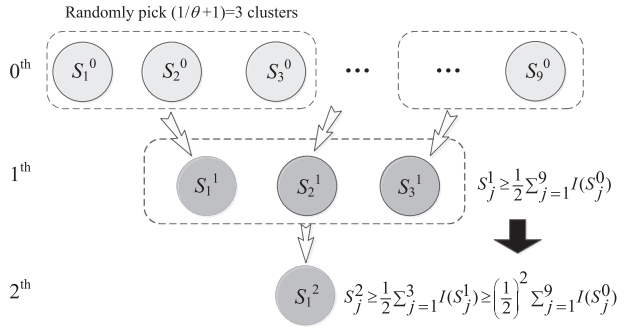


Fig. 4. An example to explain the proof of Theorem 4.2.

We note that the clusters in S^h are always θ-partitions since each cluster is recursively merged from S^{h-1} and the clusters in S^0 are θ-partitions. Thus, according to Lemma 4.1, we have the invariant $I(S_j^h) \geq 1/2 \sum_{S_x^{h-1} \in S_j^h} I(S_x^{h-1})$. The iterative process can only repeat for d times until no clusters can be merged. Intuitively, d should not exceed $\lceil \log_{(1+1/\theta)} m \rceil$ (as $k \leq m$) and thus $I(S^d) \geq \frac{1}{2}^{\lceil \log_{(1+1/\theta)} m \rceil} \sum_{i=1}^{k} I(S_i)$. Moreover, S^d only has one billboard set S_1^d, then $I(S^d) = I(S_1^d)$ and S_1^d is equal to S. Therefore, we have $I(S) \geq \frac{1}{2}^{\lceil \log_{(1+1/\theta)} m \rceil} \sum_{i=1}^{k} I(S_i)$. Moreover, as $\sum_{i=1}^{k} I(S_i) \geq I(S^*) \geq (1 - 1/e) I(OPT)$, we conclude that $I(S) \geq \frac{1}{2}^{\lceil \log_{(1+1/\theta)} m \rceil} (1 - 1/e) I(OPT)$. □

Figure 4 presents a running example to explain the iteration process in the above proof. In this example, S contains 9 clusters and $\theta = 0.5$. At iteration 0, S^0 is initialized by S; at iteration 1, each cluster of S^1 is generated by randomly merging $(1/\theta + 1 = 3)$ clusters in S^0, i.e., $S_1^1 = \{S_1^0 \cup S_2^0 \cup S_3^0\}$. According to Lemma 4.1, we have $I(S_j^1) \geq 1/2 \sum_{S_x^0 \in S_j^1} I(S_x^0)$, i.e., $I(S_1^1) \geq \frac{1}{2}(I(S_1^0) + I(S_2^0) + I(S_3^0))$. As S^1 only contains $(1/\theta + 1 = 3)$ clusters, the second iteration merges all the clusters in S^1 into one cluster S_1^2. Since $I(S_1^2) \geq 1/2 \sum_{S_x^1 \in S_j^2} I(S_x^1)$, we have $I(S_1^2) \geq 1/4 \sum_{j=1}^{9} I(S_j^0)$.

4.3 Finding a θ-partition

It is worth noting that there may exist multiple θ-partitions of U (e.g., U is a trivial θ-partition). Recall Section 4.1, the time complexity of the partition-based method (Algorithm 3) is $O(mL \cdot |\mathcal{T}| \cdot |C_m|^5)$, where $|C_m|$ is the size of the largest cluster in a partition P. Therefore, $|C_m|$ is an indicator of how good a θ-partition is, and we want to minimize $|C_m|$. Unfortunately, finding a good θ-partition is not trivial, since it can be modeled as the balanced k-cut problem where each vertex in the graph is a billboard and each edge denotes two billboards with influence overlap, which is found to be NP-hard [24]. Therefore, we use an approximate θ-partition by employing a hierarchical clustering algorithm [21]. It first initializes each billboard as its own cluster, then it iteratively merges these two clusters into one, if their overlap ratio (Equation (6)) is larger than θ. That is, for each pair of clusters $C_i, C_j \subseteq U$, if ϑ_{ij} is larger than θ, then C_i and C_j will be merged. By repeating this process, an approximate θ-partition is obtained when no cluster in U can be merged.

Note that how to efficiently get a θ partition is not the key point of this article and it can be processed offline; while our focus is how to find the influential billboards based on a θ-partition.

5 LAZY PROBE

We first propose our lazy probe algorithm to reduce the number of calls to $EnumSel(C_i, l)$ in Section 5.1, and then establish the theoretical equivalence on approximation ratio between LazyProbe and EnumSel in Section 5.2.

ALGORITHM 4: LazyProbe(P, L)

4.1 **Input:** A θ-partition P of U, budget L
4.2 **Output:** A billboard set S
4.3 Initialize two matrices $\mathbb{I}$ and ξ
4.4 **for** $i = 1$ *to* m **do**
4.5 **for** $l = 1$ *to* L **do**
4.6 $\mathbb{I}^{\downarrow}[i][l] \leftarrow \mathbb{I}[i-1][l]$
4.7 **for** $q = 1$ *to* l **do**
4.8 $\xi^{\uparrow}[i][q] \leftarrow$ **EstimateBound**(C_i, q)
4.9 **if** $\mathbb{I}^{\downarrow}[i][l] \leq \mathbb{I}[i-1][l-q] + \xi^{\uparrow}[i][q]$ **then**
4.10 **if** $\xi[i][q]$ *has not been computed* **then**
4.11 Invoke **EnumSel**(C_i, q) to compute $\xi[i][q]$
4.12 **Update** $\mathbb{I}^{\downarrow}[i][l]$ **by** $\mathbb{I}[i-1][l-q] + \xi[i][q]$
4.13 **else**
4.14 contiune;
4.15 $\mathbb{I}[i][l] \leftarrow \mathbb{I}^{\downarrow}[i][l]$
4.16 $S \leftarrow$ the corresponding selected set of $\mathbb{I}[i][l]$
4.17 **return** S

5.1 The Lazy Probe Algorithm

Recall that $\mathbb{I}[i][l]$ is the maximum influence value that can be attained with a budget not exceeding l using up to the first i clusters (in Section 4.1), and all clusters are processed in an order of their size (from the smallest to the largest by Definition 4.1). As mentioned in Section 4, $\mathbb{I}[i][l] = \max_{0 \leq q \leq l}(\mathbb{I}[i-1][l-q] + \xi[i][q])$, we need to find a q ($0 \leq q \leq l$) to maximize this influence. Note that $\mathbb{I}[i-1][l-q]$ can be easily gotten in the previous computation, but it is expensive to compute $\xi[i][q]$ by calling algorithm EnumSel. To address this issue, instead of computing the exact influence $\xi[i][q]$ in cluster C_i, we can estimate an upper bound of $\xi[i][q]$ for $0 \leq q \leq l$ (denoted by $\xi^{\uparrow}[i][q]$), and then prune the q that cannot get larger influence by bound comparison.

Algorithm 4 describes how our method works. Similar to PartSel, we employ a dynamic programming approach to compute the selected billboard set and its influence value for each cluster i and each cost l. However, the difference is that we first compute the lower bound $\mathbb{I}^{\downarrow}[i][l]$ of $\mathbb{I}[i][l]$. Obviously $\mathbb{I}^{\downarrow}[i][l] = \mathbb{I}[i-1][l]$ is a naive lower bound by setting $q = 0$ (line 4.6). Initially, when $i = 1$, for all $l \leq L$, we have $\mathbb{I}^{\downarrow}[i-1][l] = \mathbb{I}[0][l] = 0$. Then we compute an upper bound $\xi^{\uparrow}[i][q]$ from $q = 0$ to $q = l$ by calling function `EstimateBound`, which will be discussed later. Next if $\mathbb{I}^{\downarrow}[i][l] \geq \mathbb{I}[i-1][l-q] + \xi^{\uparrow}[i][q]$, we do not need to compute $\xi[i][q]$, because we cannot increase the influence using cluster C_i, and thus we can save the cost of calling EnumSel (lines 4.13–4.14). If $\mathbb{I}^{\downarrow}[i][l] < \mathbb{I}[i-1][l-q] + \xi^{\uparrow}[i][q]$, we need to compute $\xi[i][q]$, by calling $EnumSel(C_i, q)$, and update $\mathbb{I}^{\downarrow}[i][l] = \mathbb{I}[i-1][l-q] + \xi[i][q]$ (lines 4.9–4.12). Finally, we set $\mathbb{I}[i][l]$ as $\mathbb{I}^{\downarrow}[i][l]$ since we already know $\mathbb{I}^{\downarrow}[i][l]$ is good enough to obtain the solution with a guaranteed approximation ratio (line 4.16), and return the corresponding selected billboard set as S (line 4.17).

Estimation of Upper Bound $\xi^{\uparrow}[i][q]$. A key challenge to ensure the approximation ratio of LazyProbe is to get a tight upper bound $\xi^{\uparrow}[i][q]$. Unfortunately, we observe that it is hard to obtain a tight upper bound efficiently due to the overlap influence among billboards. Fortunately, we can get an approximate bound, with which our algorithm can still guarantee the $(1 - 1/e)$ approximation ratio (see Section 5.2). To achieve this goal, we first utilize the basic greedy algorithm GreedySel to select the billboards $S' = \{b_1, b_2, \ldots, b_k\}$. Let b_{k+1} be the next billboard with the

FUNCTION 5: EstimateBound(U, L)

5.1 **Input:** A billboard set U, a budget L
5.2 **Output:** An influence estimator $\xi^{\uparrow}[i][q]$
5.3 S' = GreedySel (U, L, ϕ)
5.4 b_{k+1} is the next billboard with the largest unit marginal influence
5.5 $\xi^{\uparrow}[i][q] = I(S') + [L - w(S')]\frac{\Delta(b_{k+1}|S')}{w(b_{k+1})}$
5.6 **return** $\xi^{\uparrow}[i][q]$

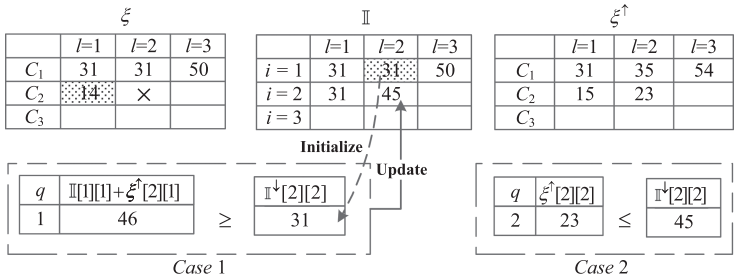


Fig. 5. A running example for LazyProbe.

maximal marginal influence. If we include b_{k+1} in the selected billboards, then the cost will exceed L. If we do not include it, we will lost the cost of $L - w(S')$ where $w(S') = \sum_{1 \le i \le k} w(b_i)$. Then we can utilize the unit marginal influence of b_{k+1} to remedy the lost cost, and thereby we can get an upper bound as $\xi^{\uparrow}[i][q] = I(S') + [L - w(S')]\frac{\Delta(b_{k+1}|S')}{w(b_{k+1})}$. We later show that $\xi^{\uparrow}[i][q] \ge (1 - 1/e)\xi[i][q]$. Moreover, the solution quality of Algorithm 4 remains the same as Algorithm 3. The details of the theoretical analysis will be presented in Section 5.2.

Example 5.1. Figure 5 shows an example on how Algorithm 4 works. There are three clusters C_1, C_2, and C_3 in a partition P, and the estimator matrix $\xi^{\uparrow}$ is shown in upper right corner. When C_i $(i = 1, 2, 3)$ is considered, it computes $\mathbb{I}[i][l]$, for each $l = 1, \ldots, L$, by the bound comparisons. Taking $\mathbb{I}[2][2]$ as an example, Algorithm 4 first initializes $\mathbb{I}^{\downarrow}[2][2] = \mathbb{I}[1][2]$ and then computes $\mathbb{I}[1][2 - q] + \xi^{\uparrow}[2][q]$ $(q = 1, 2)$ for bound comparisons. For Case 1 $(q = 1)$, as $\mathbb{I}^{\downarrow}[2][2] \le \mathbb{I}^{\uparrow}[2][2]$, Algorithm 4 needs to compute $\xi[2][1]$ by invoking EnumSel and update $\mathbb{I}^{\downarrow}[2][2]$ by $\mathbb{I}[1][1] + \xi[2][1]$. For Case 2 $(q = 2)$, since $\mathbb{I}^{\downarrow}[2][2] \ge \mathbb{I}^{\uparrow}[2][2]$, $\mathbb{I}^{\downarrow}[2][2]$ does not need to be updated and finally $\mathbb{I}[2][2] = \mathbb{I}^{\downarrow}[2][2] = 45$.

The complexity of LazyProbe is the same as that of PartSel in the worst case, but the pruning strategy actually can work well and reduce the running time greatly (as evidenced in our experimental study of Section 7).

5.2 Theoretical Analysis

In this section, we conduct theoretical analysis to establish the equivalence between LazyProbe and PartSel in terms of the approximation ratio. We first show that if the bound $\xi^{\uparrow}[i][l]$ in LazyProbe is $(1 - 1/e)$ approximate to *TIP* instance of billboards in cluster i using budget l, then the approximation ratio of *LazyProbe* and *PartSel* is the same (Theorem 5.1). We then move on to show that $\xi^{\uparrow}[i][l]$ is indeed $(1 - 1/e)$-approximate in Lemmas 5.2–5.4.

Theorem 5.1. *If $\xi^{\uparrow}[i][l]$ obtained by Algorithm 5 achieves a $(1-1/e)$ approximation ratio to the TIP instance for cluster i with budget l, LazyProbe ensures the same approximation ratio with PartSel presented in Section 4.1.*

Proof. Let C_i denote the the ith cluster considered by LazyProbe and $U_i = \bigcup_{j=1}^{i} C_j$. To prove the correctness of this theorem, we first prove $\mathbb{I}[i][l] \geq (1-1/e)I(OPT^l_{U_i})$ for all $i \leq m$ and $l \leq L$, where $OPT^l_{U_i}$ is the optimal solution of the TIP instance for billboard set U_i with budget l. Clearly, if this assumption holds, we have $\sum_{i=1}^{k} I(S_i) \geq (1-1/e)I(OPT)$ (OPT is the globally optimal solution). $S = \{S_1, \ldots, S_k\}$ is the solution returned by LazyProbe. $S_i = S \cap C_i$.

We prove it by mathematical induction. When $i = 0$, this assumption holds immediately. When $i > 0$, we assume that the assumption holds for the first ith recursion, and prove it still holds for the $(i+1)$th recursion. According to the definition, we have $\xi^{\uparrow}[i+1][l] \geq (1-1/e)I(OPT^l_{C_{i+1}})$. Moreover, we have already assumed $\mathbb{I}[i][l] \geq (1-1/e)I(OPT^l_{U_i})$ $(l = 1, \ldots, L)$, thus $\mathbb{I}[i+1][l] = \max_{0\leq q\leq l}\{\mathbb{I}[i][l-q] + \xi^{\uparrow}[i+1][q]\} \geq (1-1/e)\max_{0\leq q\leq l}\{I(OPT^{l-q}_{U_i}) + I(OPT^{q}_{C_{i+1}})\}$. Moreover, as $\max_{0\leq q\leq l}\{I(OPT^{l-q}_{U_i}) + I(OPT^{q}_{C_{i+1}})\} \geq I(OPT^{l}_{U_{i+1}})$, we have $\mathbb{I}[i+1][l] \geq (1-1/e)I(OPT^l_{U_{i+1}})$. The assumption gets proof.

Since S comes from k clusters and $k \leq m$, we can conclude that $I(S) \leq \frac{1}{2}^{\lceil \log_{(1+1/\theta)} m \rceil} \sum_{i=1}^{k} I(S_i)$. We omit to prove this conclusion here since the proof is similar to that of Theorem 4.2. Moreover, as $\sum_{i=1}^{k} I(S_i) \geq (1-1/e)I(OPT)$ (the assumption holds), thus $I(S) \leq \frac{1}{2}^{\lceil \log_{(1+1/\theta)} m \rceil}(1-1/e)I(OPT)$. □

Theorem 5.1 requires that $\xi^{\uparrow}[i][l]$ is $(1-1/e)$-approximate to the corresponding TIP instance in a small cluster. To show that $\xi^{\uparrow}[i][l]$ returned by Algorithm 5 satisfies such requirement, we describe the following hypothetical scenario for running Algorithm 1 on the TIP instance for cluster C_i and budget l. Let b_{k^*+1} be a billboard in the optimal solution set, and it is the first billboard that violates the budget constraint in Algorithm 1. The following inequality holds [13].

Lemma 5.2 [13]. *After the ith iteration ($i = 1, \ldots, k^*+1$) of the hypothetical scenario running Algorithm 1, the following holds:*

$$I(S_i) \geq \left[1 - \prod_{j=1}^{i}\left(1 - \frac{w(\{b_j\})}{L}\right)\right] \cdot I(OPT) \tag{7}$$

Where S_i be the billboard set that is selected by the first i iterations of the hypothetical scenario.

With Lemma 5.2, we analyze the solution quality of running the hypothetical scenario by using the k^*+1 billboards to deduce $\xi^{\uparrow}[i][l]$.

Lemma 5.3. *Let $\mathcal{M}_{k^*+1}$ denote the unit marginal influence of adding b_{k^*+1} in the hypothetical scenario, i.e., $\mathcal{M}_{k^*+1} = [I(S_{k^*} \cup \{b_{k^*+1}\}) - I(S_{k^*})]/w(\{b_{k^*+1}\})$. Then $I(S_{k^*}) + [L - w(S_{k^*})] \cdot \mathcal{M}_{k^*+1} \geq (1-1/e)I(OPT)$.*

Proof. First, we observe that for $a_1, \ldots, a_n \in \mathbb{R}^+$ such that $\sum_{i=1}^{n} a_i = \alpha A$, the function

$$1 - \prod_{i=1}^{n}\left(1 - \frac{a_i}{A}\right)$$

achieves its minimum of $1-(1-\alpha/n)^n$ when $a_1 = a_2 = \cdots = a_n = \alpha A/n$.

Suppose b' is a virtual billboard with cost $L - w(S_{k^*})$ and the unit marginal influence of b' to S_i is $\mathcal{M}_{k^*+1}$, for $i = 1, \ldots, k^*$. We modify the instance by adding b' into U and let $U \cup \{b'\}$ be denoted

by U'. Then after the first k^*th iterations of Algorithm 1 on this new instance, b' must be selected at the (k^*+1)th iteration. As $L(S_{k^*}) + w(\{b'\}) = L$, by applying Lemma 5.2 and the observation to $I(S')$ ($S' = S_k \cup \{b'\}$), we get:

$$\begin{aligned} I((S') &\geq \left[1 - \prod_{j=1}^{k^*+1}\left(1 - \frac{w(\{b_j\})}{L}\right)\right] \cdot I(OPT') \\ &\geq \left(1 - \left(1 - \frac{1}{k^*+1}\right)^{k^*+1}\right) \cdot I(OPT') \\ &\geq \left(1 - \frac{1}{e}\right) \cdot I(OPT') \end{aligned}$$

Note that $I(OPT')$ is surely larger than $I(OPT)$, thus $I(S_{k^*}) + [L - w(S_{k^*})]\mathcal{M}_{k^*+1} = I(S') \geq (1 - \frac{1}{e}) \cdot I(OPT') \geq (1 - \frac{1}{e}) \cdot I(OPT)$. □

Finally, we show that the estimator obtained by Algorithm 5 is larger than the bound value obtained by the hypothetical scenario described in Lemma 5.3, which indicates that Algorithm 5 is $(1 - 1/e)$-approximate and it further implies Theorem 5.1 hold.

Lemma 5.4. *Given an instance of TIP. Let $\xi[i][l]$ be an estimator returned by Algorithm 5, we have $\xi[i][l] \geq (1 - 1/e)I(OPT)$.*

Proof. We observe that $\mathcal{M}_{k^*+1}$ cannot be larger than $\mathcal{M}_{k+1}$ and $I(S_k) + (L - w(S_k))\mathcal{M}_{k+1} \geq I(S_{k^*}) + (L - w(S_{k^*}))\mathcal{M}_{k^*+1}$. Moreover, Lemma 5.3 shows $I(S_{k^*}) + (L - w(S_{k^*}))\mathcal{M}_{k^*+1} \geq (1 - 1/e)I(OPT)$, so $I(S_k) + (L - w(S_k))\mathcal{M}_{k+1} \geq (1 - 1/e)I(OPT)$. As $\xi[i][l] = I(S) + [L - w(S)]\mathcal{M}_{k+1}$ (Algorithm 5 line 5.5), this lemma is proved. □

6 FASTENUM

To further improve the efficiency, this section introduces an optimization framework FastEnum to reduce the costs of the enumeration and the marginal influence computation for the above proposed methods, i.e., EnumSel, PartSel, and LazyProbe. It mainly consists of two parts: an EnumSel early termination and an aggregated trajectory index.

6.1 EnumSel Early Termination

LazyProbe accelerates PartSel by reducing the invocations of EnumSel. However, the runtime of EnumSel still is a major bottleneck of the partition-based framework because it needs to enumerate all the feasible size-3 sets about U (see the computation of H_2 in Algorithm 2, lines 2.5–2.9). To overcome this issue, we introduce a branch-and-bound strategy to speed up EnumSel. The core idea is to skip the enumerations on those billboards that would not be in the answer set, and thus accelerate the computation of H_2. Intuitively, if U can be skipped to one half, the runtime of EnumSel should be reduced greatly due to the complexity of $O(|\mathcal{T}||U|^5)$. To this end, we establish an upper bound for all the solutions that contain b. Clearly, if this bound is worse than the current solution, b can be eliminated from U since it is impossible for any solution with b to improve the final influence. As mentioned in Section 5.1, finding a tight upper bound for a given set is not trivial. Thereby, we turn to compute an approximate bound for b as below.

Definition 6.1. Given a billboard $b \in U$, let F_b denote the set of feasible solutions containing b. If $\nexists f_b \in F_b$ such that $(1 - 1/e)I(f_b) > \mathcal{H}(b)$ holds, we say that $\mathcal{H}(b)$ is an approximate bound of F_b.

The following lemma shows that EstimateBound (see Section 5.1) can be utilized to obtain $\mathcal{H}(b)$ if we rewrite the line 5.3 of Function 5 by '$S' = GreedySel(U, L, b)$'.

Lemma 6.1. *Given $b \in U$, Function 5 with the initialized seed b, denoted by $EstimateBound(U, L, b)$, returns an approximate bound $\mathcal{H}(b)$ of F_b.*

Proof. To show the correctness, we generate a hypothetical instance by eliminating b and its influenced trajectories from the original instance, w.r.t. $U - \{b\}, \mathcal{T} - \mathcal{T}_{\{b\}}$. Let us denote the influence of the optimal solution for this instance as OPT'. According to Lemma 5.4, $EstimateBound(U', L)$ returns a bound larger than $(1 - 1/e)OPT'$, and thus the returned value of $EstimateBound(U, L, b)$ would not be less than $I(b) + (1 - 1/e)OPT'$. As $I(f_b) \leq I(b) + OPT'$ for $\forall f_b \in F_b$ and $I(b) + (1 - 1/e)OPT' \geq (1 - 1/e)(I(b) + OPT')$. We can then conclude that $EstimateBound(U, L, b)$ must return an approximate bound $\mathcal{H}(b)$ of F_b. □

According to Lemma 6.1, our optimized method uses Function 6 to boost EnumSel to replace the lines 2.5–2.9 of Algorithm 2. In each iteration, the function deletes b from U if its bound $\mathcal{H}(b)$ is smaller than the current influence. Otherwise, it enumerates all size-3 sets with b and complements each of them greedily. At last, it updates H_2 for the next iteration if a more influential set is found during the enumeration process.

The following theorem shows that this modification would not be inferior to the original version in effectiveness, and still can return $(1 - 1/e)$-approximation for TIP.

Theorem 6.2. *By applying Function 5 to EnumSel, this modified EnumSel returns $(1 - 1/e)$-approximation for TIP.*

Proof. Suppose S^* is the optimal solution of TIP. According to Definition 6.1, we have $\max[I(f_b)] = OPT$ and $\mathcal{H}(b) > (1 - 1/e)OPT$, for $\forall b \in S^*$. Intuitively, Function 5 can eliminate any $b \in S^*$ from U only if the current solution is more influential than $\mathcal{H}(b)$. At this case, the algorithm returns a $(1 - 1/e)$-approximation no doubt.

The rest case is that there has no $b \in S^*$ can be pruned. We denote the eliminated set as U'. Clearly, S^* is the optimal solution for the instance of $U - U'$, which implies $OPT' = OPT$. On this new instance, the modified algorithm is degenerated to EnumSel since no billboard in $U - U'$ can be skipped. Therefore, the algorithm obtains a solution with influence greater than $(1 - 1/e)OPT'$ and thus it is a $(1 - 1/e)$-approximation for TIP. □

Moreover, by making use of the submodularity property, we can use the Cost-Effective Lazy Forward strategy [14] to reduce the number of $\Delta(b|S)$ computations. In particular, the main idea behind is that the marginal influence of a candidate in the current iteration should not be more than that in previous iterations, and thus the $\Delta(b|S)$ computations can be greatly pruned. For example,

FUNCTION 6: FastEnum(U, L, b)

```
6.1  Input: A billboard set U, a budget L and a billboard b
6.2  Output: Enumeration result H_2
6.3  H_2 ← φ
6.4  for all b ∈ U, such that w(b) ≤ L do
6.5  |  H(b) ← EstimateBound(U, L, b)
6.6  for all b ∈ U, such that w(b) ≤ L do
6.7  |  if I(H_2) ≥ H(b) then
6.8  |  |  U ← U − b
6.9  |  |  Continue
6.10 |  Compute the influence of all the size-3 sets with b
6.11 |  Update H_2 if we find that a set is better than current optimal solution
6.12 return H_2;
```

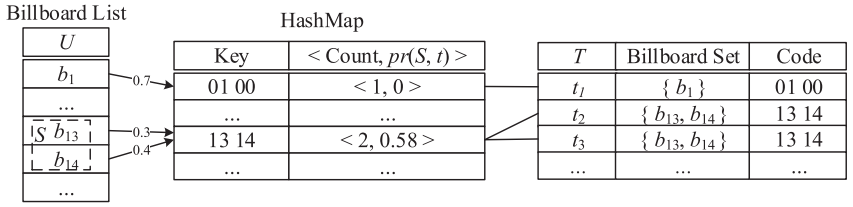


Fig. 6. An example of Hash-based aggregation. Suppose $X = 2$ and $Y = 2$.

$I(b')$ is an upper-bound of $\Delta(b'|S)$. If $\Delta(b|S)$ is larger than $I(b')$, we can prune b' safely because even its upper bound is not desired.

6.2 Aggregated Trajectory Index

The most expensive part of all the proposed algorithms above is to compute $\Delta(b|S)$ (see line 1.4 of Algorithm 1), which in turn transforms to the computation of $pr(S, t)$ and $pr(S \cup \{b\}, t)$ for each trajectory t in $\mathcal{T}$. However, we observe that many trajectories in real world are very similar. It enables us to reduce the size of $\mathcal{T}$ by aggregating these "repetitive" trajectories that pass the same billboards. Therefore, in this section, we first show how to aggregate trajectories, and then propose an effective mapping index based on the aggregation result to accelerate our algorithms.

Hash-based aggregation. Intuitively, the trajectory aggregation by brute-force should be very costly as it needs to compare the billboard sets of each pair of trajectories in $\mathcal{T}$. To overcome this issue, we introduce a hash-based aggregation which only takes a linear time to generate the result.

The core idea is to encode each trajectory according to its passed billboards' *id*, and group the trajectories with the same code by a hash aggregation. Without loss of generality, we suppose that any trajectory only can pass X billboards at most, and the maximum *id* of billboards can be represented by a Y-digit number. Given a billboard set $\{b_1, \ldots, b_i\}$ (ordered by *id* ASC) of t, t is encoded by a $X \cdot Y$ length code as follows. (1) We encode each billboard in U as a Y-digit code according to its *id*. As shown in the left of Figure 6, b_1 is encoded by "01" and b_{14} is encoded by "14." (2) Based on U's codebook, we join the *id*s of billboards passed by t as the code of t, i.e., t_2 is encoded by "13 14," since t passes b_{13} and b_{14}. Note that, if the number of the passed billboards of t is less than X, we complement the rest of digits by "0," i.e., t_1 is encoded by "01 00." In this manner, the "repetitive" trajectories are encoded by the same code. To efficiently aggregate them together, a HashMap is used to distribute trajectories across an array of buckets, which is shown as the middle of Figure 6. In this figure, each bucket represents a cluster of trajectories that pass the same billboards, in which we use a pair of variables $<count, pr(S, t)>$ to record the number of trajectories and the influence on any trajectory in the bucket.

Intuitively, given a billboard set S, we can deal with the individuals in a bucket as a whole, and compute the influence of S to them by utilizing the values of $<count, pr(S, t)>$. Taking Figure 6 as an example, the influence of S to the bucket "13 14" can be computed by $count \times pr(S, t) = 2 \times 0.58$. Followed by this idea, we propose an aggregated index based on HashMap to boost the computation of $\Delta(b|S)$.

Aggregated index. The index is shown in Figure 6. In this figure, we link a billboard to a bucket by a pointer, if the billboard can influence the trajectories in the bucket. Given a set S and a candidate b, we can obtain the influenced trajectories by scanning the buckets which are linked to $S \cup \{b\}$, and compute $\Delta(b|S)$ by summing up the marginal influence on each influenced buckets. Here, the marginal influence on a bucket equals $count \times [pr(S \cup \{b\}, t) - pr(S, t)]$. As $pr(S, t)$ is obtained, the key is how to compute the influence on t when b is added into S, w.r.t. $pr(S \cup \{b\}, t)$. Recall

Table 3. Statistics of Datasets

	$\|\mathcal{T}\|$	$\|U\|$	AvgDistance	AvgTravelTime	AvgPoint#
NYC	4 m	1,500	2.9 km	569 s	159
LA	200 k	2,500	2.7 km	511 s	138

Equation (1), this computation needs to union the trajectories influenced by each billboard $S \cup \{b\}$ and thus takes $O(|S|)$ time. However, the following equation shows that $pr(S \cup \{b\}, t)$ can be obtained by simple a union of $pr(S, t)$ and $pr(b, t)$ directly.

$$pr(S \cup \{b\}, t) = 1 - (1 - pr(S, t))(1 - pr(b, t)) \tag{8}$$

Note that, the computation of Equation (8) only takes $O(1)$ time and will be much more efficient than that of Equation (1) when $|S|$ is large.

7 EXPERIMENTS

In this section, we evaluate our proposed optimization techniques with extensive experiments on real-world datasets. We conduct experimental evaluation on the effectiveness, efficiency, and scalability of our solutions. We first introduce the experimental setup, and then report our findings.

7.1 Experimental Setup

Datasets. We collect billboards and trajectories data for the two largest cities in US, i.e., NYC and LA.

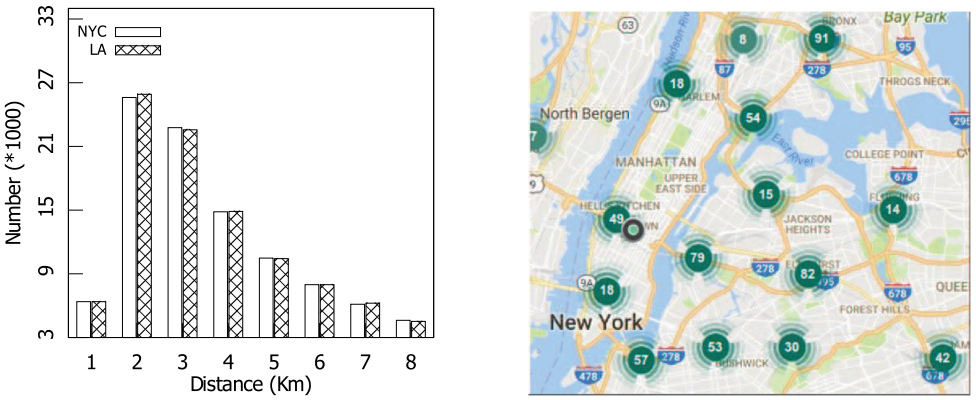
Billboard data is crawled from LAMAR,[1] one of the largest outdoor advertising companies worldwide.

Trajectory data is obtained from two types of real datasets: the TLC trip record dataset[2] for NYC and the Foursquare check-in dataset[3] for LA. For NYC, we collect TLC trip record containing green taxi trips from Jan 2013 to Sep 2016. Each individual trip record includes the pick-up and drop-off locations, time, and trip distances. We use Google Maps API[4] to generate the trajectories, and if the distance of the recommended route by Google is close to the trip distance and travel time in the original record (within 5% error rate), we use this route as an approximation of this trip's real trajectory. As a result, we obtain 4 million trajectories for trip records as our trajectory database. For LA, as there is no public taxi record, we collect the Foursquare check-in data in LA, and generate the trajectories using Google Maps API by randomly selecting the pick-up and drop-out locations from the check-ins.

The statistics of those datasets are shown in Table 3, the distribution of trajectories' distance is shown in Figure 7(a), and a snapshot of the billboards' locations in NYC is shown in Figure 7(b). We can find that over 80% trips finish in 5 kilometers.

Algorithms. As mentioned in Sections 1 and 2.2, this is the first work that studies the TIP problem, there exists no previous work for direct comparison. In particular, we compare five methods: TrafficVol which picks billboards by a descending order of the volume of trajectories meeting those billboards within the budget L; a basic greedy GreedySel (Algorithm 1); a greedy enumeration method EnumSel (Algorithm 2); our partition based method PartSel (Algorithm 3); and our lazy

[1] http://www.lamar.com/InventoryBrowser.
[2] http://www.nyc.gov/html/tlc/html/about/trip_record_data.shtml.
[3] https://sites.google.com/site/yangdingqi/home.
[4] https://developers.google.com/maps/.



(a) Trajectory Distance (b) Billboard Location

Fig. 7. Distribution of datasets in NYC.

Table 4. Parameter Setting

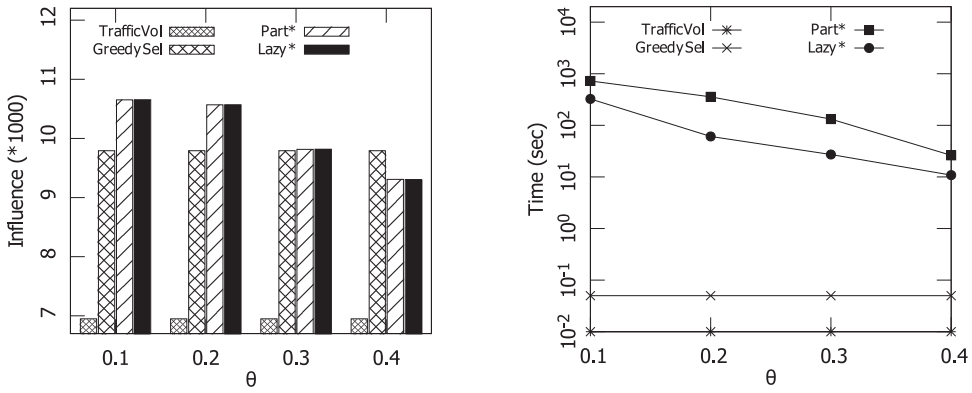
Parameter	Values
L	100k, **150k**, ... 300k
$\|\mathcal{T}\|$ (NYC)	40k, ...,**120k**..., 4m
$\|\mathcal{T}\|$ (LA)	40k, 80k,**120k**, 160k, 200k
$\|U\|$ (NYC)	0.5k, 1k, 1.46k, (**2k**...10k by replication)
$\|U\|$ (LA)	1k, **2k**, 3k, (4k... 10k by replication)
θ	0, 0.1, **0.2**, 0.3, 0.4
λ	25m, **50m**, 75m, 100m
$pr(b,t)$	**0.8**

probe method LazyProbe (Algorithm 4). Moreover, to illustrate the improvement of the optimizations proposed in Section 6, we also implement the proposed optimizations (i.e., Function 6) into EnumSel, PartSel, and LazyProbe, and denote the optimized versions as Enum*, Part*, and Lazy*, respectively. Note that EnumSel is too slow to converge in 150 minutes even for a small dataset (because the complexity of EnumSel is proportional to $|U|^5$). Thus in our default setting, we do not include EnumSel. Instead we add one experiment on a smaller dataset of NYC to evaluate it and its optimized version in Section 7.3.

Performance measurement. We evaluate the performance of all methods by the runtime and the influence value of the selected billboards. Each experiment is repeated 10 times, and the average performance is reported.

Billboard costs. Unfortunately, all advertising companies do not provide the exact leasing cost; instead, they provide a range of costs for a suburb. For example, the costs of billboards in New Jersey-Long Island by LAMAR range from 2,500 to 14,000 for 4 weeks [3]. So we generate the cost of a billboard b by designing a function proportional to the number of trajectories influenced by b: $w(b) = \lfloor \beta \cdot I(b)/100 \rfloor \times 1000$, where β is a factor chosen from 0.8 to 1.2 randomly to simulate various cost/benefit ratios. Here we compute the cost w.r.t. $|\mathcal{T}| = 200k$ trajectories.

Parameters. Table 4 shows the settings of all parameters, such as the distance threshold λ to determine the influence relationship between a trajectory and a billboard, the threshold for θ-partition, the budget L and the number of trajectories $|\mathcal{T}|$. The default one is highlighted in bold; we vary



(a) Influence (b) Efficiency

Fig. 8. Effect of varying θ on NYC.

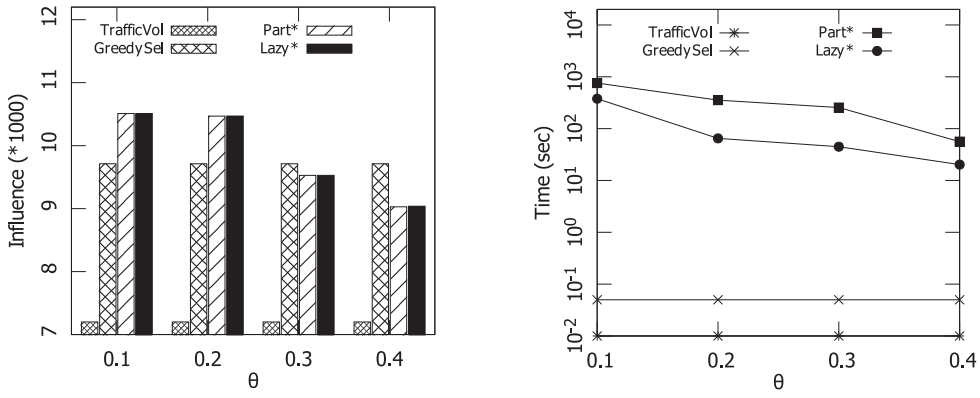
one parameter while the rest parameters are kept default in all experiments unless specified otherwise. Since the total number of real-world billboards in LAMAR is limited (see Table 3), the $|U|$ larger than the limit are replicated by random selecting locations in the two cities. To avoid the replicated billboards without influencing any trajectories, we discard the billboards that have zero influence during the generation process.

Setup. All codes are implemented in Java, and experiments are conducted on a personal computer with 1.8 GHz Intel CORE 8 Core CPU and 8 GB memory running windows/10 OS. In our experiment, the x-axis of efficiency tests are plotted in logarithm scale.

7.2 Experiments

7.2.1 Choice of θ-partition. Since θ-partition is an input of PartSel method and θ indicates the degree of overlap among clusters generated in the partition phase of PartSel (and LazyProbe), we would like to find a generally good choice of θ that strikes a balance between the efficiency and effectiveness of PartSel and LazyProbe. Note that, this study selects their optimized versions, Part* and Lazy*, to evaluate the impact of θ.

We vary θ from 0 to 0.4, and record the number of clusters as input of Part* and Lazy* methods, the percentage of the largest cluster size over U (i.e., $\frac{|C_m|}{|U|}$), the runtime, and the influence value of Part* and Lazy*. The results on both datasets are shown in Figures 8–10 and Table 5. By linking those results, we have four main observations. (1) With the increase of θ, the influence quality decreases and the efficiency is improved, because for a larger θ, the tolerated influence overlap is larger and there are many more clusters with larger overlaps. (2) When θ is 0.1 and 0.2, Part* and Lazy* achieve the best influence (Figures 8(a) and 9(a)), while the efficiency of 0.2 is not much worse than that of $\theta = 0.3$ (Figures 8(b) and 9(b)). The reason is that, it results in an appropriate number of clusters (e.g., 23 clusters for NYC dataset at $\theta = 0.2$ in Figure 10), and the largest cluster covers 7.1% of all billboards, as evidenced by the value of $\frac{|C_m|}{|U|}$ in Table 5. (3) In one extreme case that $\theta = 0.4$, although the generated clusters are dispersed and small, it results in high overlaps among clusters, so the influence value drops and becomes worse than GreedySel, and meanwhile the efficiency of Part* (Lazy*) only improves by around 12 (6) times as compared to that of $\theta = 0.2$ on the NYC (LA) dataset. The reason is that Part* and Lazy* find influential billboards within a cluster and do not consider the influence overlap to billboards in other cluster; thus when θ is large, high overlaps between the clusters lead to a low precision of Part* and Lazy*. (4) All other methods beat the TrafficVol baseline by 45% in term of the influence value of selected billboards.



(a) Influence (b) Efficiency

Fig. 9. Effect of varying θ on LA.

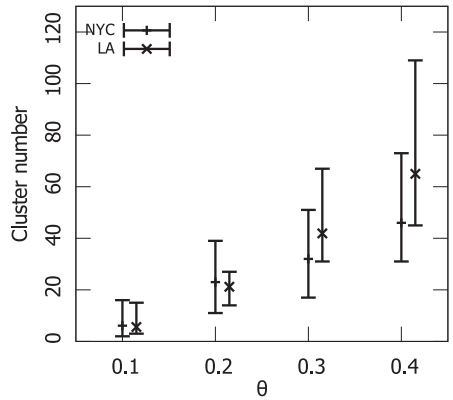


Fig. 10. θ vs Number of clusters.

Table 5. $|C_m|/|U|$, w.r.t. Varying θ

	0.1	0.2	0.3	0.4
NYC	12.6%	7.1%	6.4%	5.8%
LA	13.5%	7.8%	5.9%	5.1%

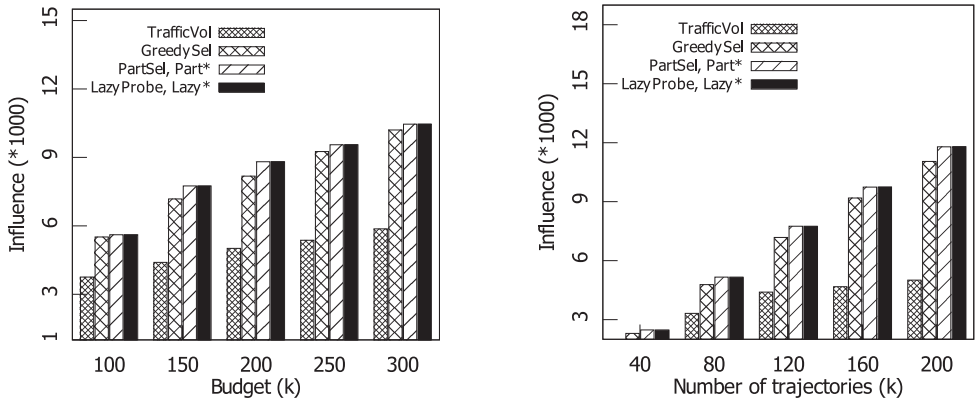
The result on LA is very similar to that of NYC, so we omit the description here. Therefore, we choose 0.2 as the default value of θ in the rest of the experiments.

7.2.2 Choice of Influence Metric. Recall Section 2.1 that the influence of a billboard b_i to a trajectory t_j, $pr(b_i, t_j)$, is defined. Here we conduct more experiments to test the impact of an alternative choice for the influence probability measurement as described in Section 7. The alternative choice is: $pr(b_i, t_j) = b_i.panelsize/(2 * maxPanelSize)$ where $maxPanelSize$ is the size of the largest billboard in U, and we further normalize by 2 to avoid a too large probability, say 1.

The influence result of all algorithms on the NYC dataset is shown in Figure 11. Recall our corresponding experiment of adopting choice 1 in Figures 14(a) and 16(a), we have the same observations: PartSel and LazyProbe outperform all the rest algorithms in influence. To summarize, our solutions are orthogonal to the choice of these metrics.

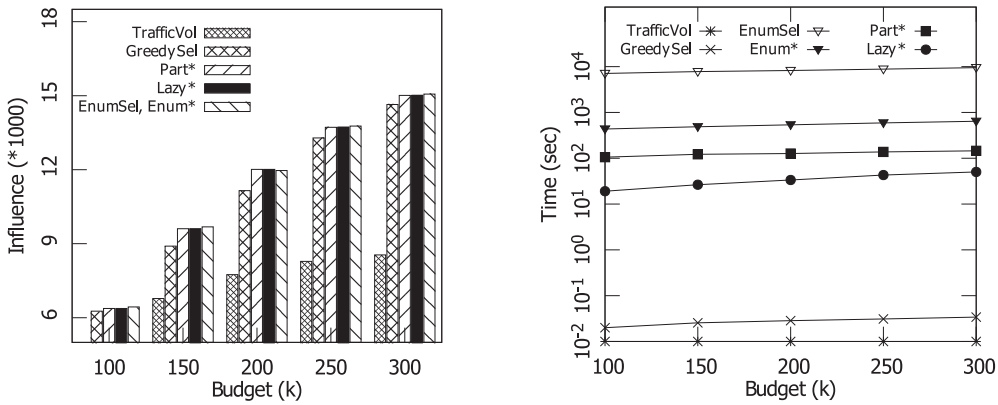
7.3 Evaluation Against EnumSel on Small Datasets

As reported in Section 7.1, EnumSel could not terminate in a reasonable time for most experiments' default settings due to its dramatically high computation cost $O(|\mathcal{T}| \cdot |U|^5)$. Therefore, we generate a small subset of the NYC dataset to ensure that it can complete in reasonable time, and compare



(a) Influence (Varying L) (b) Influence (Varying $|\mathcal{T}|$)

Fig. 11. Test on other choice of influence probability.



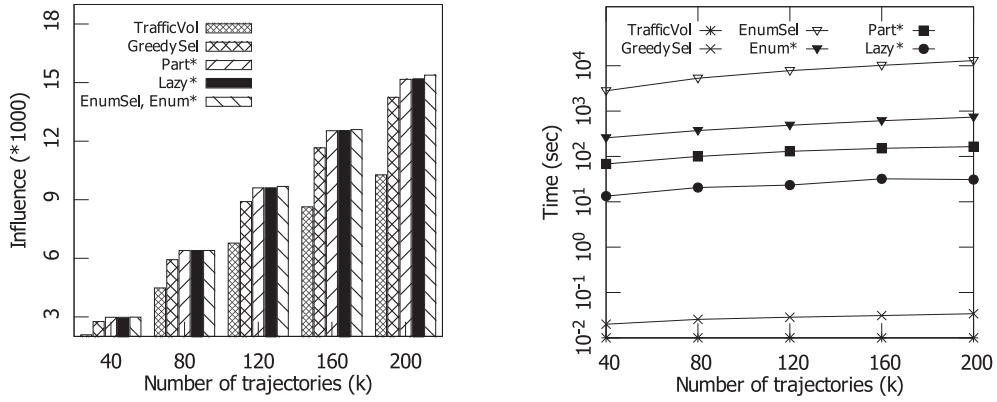
(a) Influence (L) (b) Efficiency (L)

Fig. 12. Testing EnumSel on a small NYC dataset ($|U|$ = 1,000, $|\mathcal{T}|$ = 120k).

its performance with other approaches proposed in this article. In particular, we have the default setting of $|U|$ = 1,000 and $|\mathcal{T}|$ = 120k.

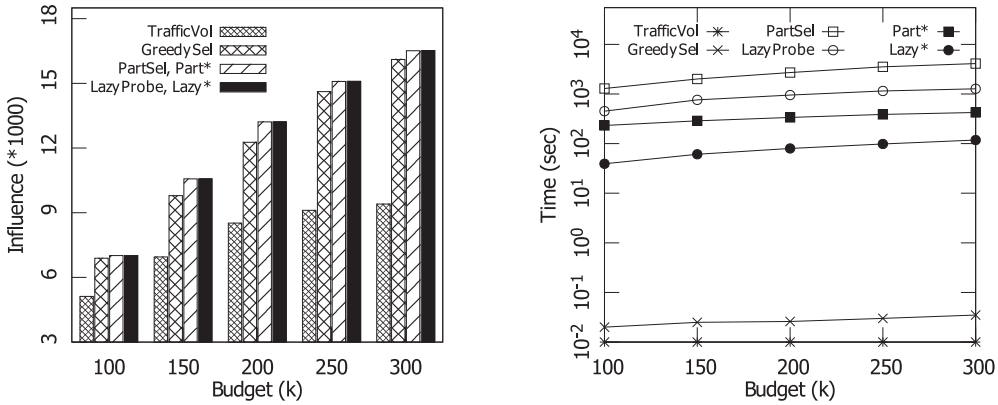
Figures 12(a) and 13(a) show the effectiveness of all algorithms when varying the budget and the number of trajectories respectively. From Figure 12(a) we make three observations: (1) When L is small, the influence of Enum* (EnumSel) is better than that of Part* and Lazy*. It is because when only a small number of billboards can be afforded, the enumerations can easily find the optimal set since the possible world of feasible sets is small, whereas Part* and Lazy* are mainly obstructed by reduplicating the influence overlaps between clusters. (2) With the growth of L, the advantage of Enum* gradually drops, while Part* and Lazy* achieve better influence; and when the budget reaches 200k, they have almost the same influence as Enum*.

The efficiency results are presented in Figures 12(b) and 13(b), w.r.t. a varying budget and trajectory number. We find: (1) Part* and Lazy* consistently beat Enum* by almost two and one order of magnitude, respectively. (2) EnumSel has the worst performance among all algorithms and is slower than Enum* more than one order of magnitude. This is because that Enum* can prune a



(a) Influence ($|\mathcal{T}|$) (b) Efficiency ($|\mathcal{T}|$)

Fig. 13. Testing EnumSel on a small NYC dataset ($|U|$ = 1,000, L= 150k).



(a) Influence (b) Efficiency

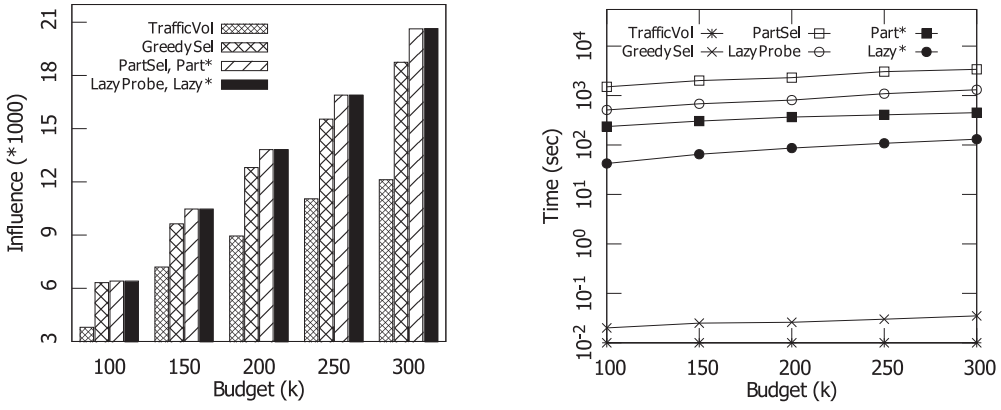
Fig. 14. Effect of varying budget L on NYC.

large number of enumerations via eliminating insignificant candidates from U, while avoiding to compute the marginal gain on each single trajectory in $\mathcal{T}$.

7.4 Effectiveness Study

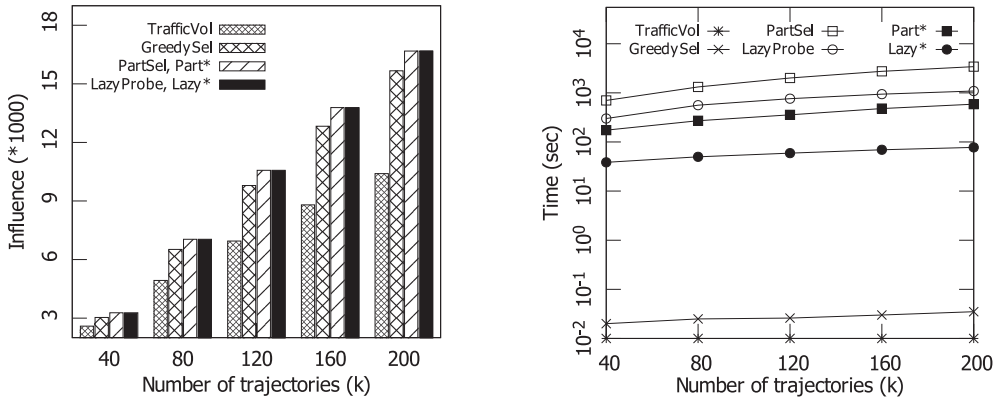
We study how the influence is affected by varying the budget L, the trajectory number $|\mathcal{T}|$, the distance threshold λ, and the overlap ratio, respectively. Lastly, we study the approximation ratio of all algorithms.

7.4.1 Varying the Budget L. The influence of all algorithms on NYC and LA by varying the L is shown in Figures 14 and 15, and we have the following observations on both datasets. (1) TrafficVol has the worst performance. PartSel (Part*) and LazyProbe (Lazy*) achieve the same influence and their improvement over TrafficVol exceeds 99%. (2) With the growth of L, the advantage of PartSel and LazyProbe over GreedySel is increasing, from 1.8% to 6.5% when L varies from 100k to 300k on LA dataset. This is because when the influence overlaps between clusters, it cannot be avoided, then how to maximize the benefit/cost ratio in clusters is critical to enhance the performance,



(a) Influence (b) Efficiency

Fig. 15. Effect of varying budget L on LA.



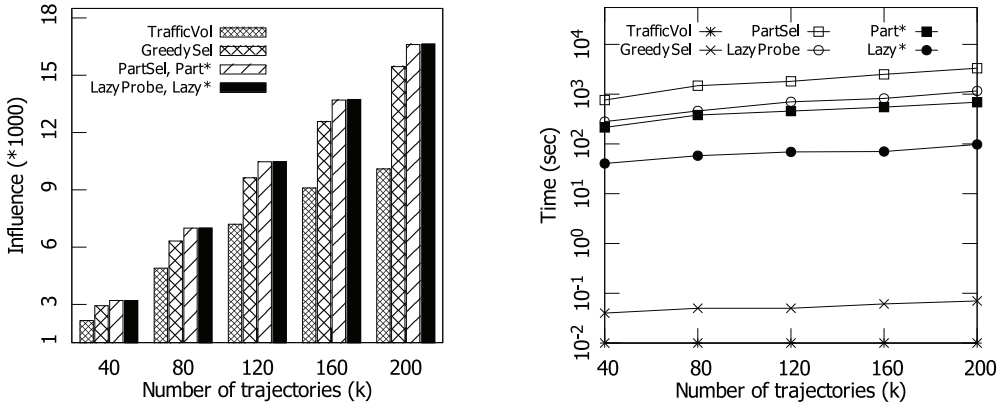
(a) Influence (b) Efficiency

Fig. 16. Effect of varying trajectory number $|\mathcal{T}|$ on NYC.

which is exactly achieved by PartSel and LazyProbe, since they exploit the locality feature within clusters.

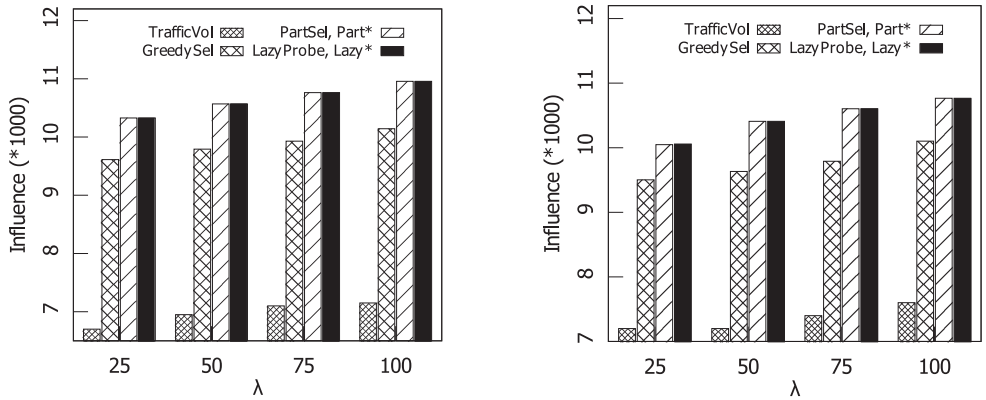
7.4.2 Varying the Trajectory Number $|\mathcal{T}|$. Figures 16 and 17 show the result by varying $|\mathcal{T}|$. We find: (1) the influence of all methods increases because more trajectories can be influenced; and (2) the influence by PartSel and LazyProbe is consistently better than that of GreedySel and TrafficVol, because the trajectory locality is an important factor that should be considered to increase the influence.

7.4.3 Varying λ. Figure 18 shows the influence result by varying the threshold λ, which determines the influence relationship between billboards and trajectories (in Definition 2.1). We make two observations. (1) With the increase of λ, the performance of all algorithms becomes better, because a single billboard can influence more trajectories. (2) PartSel and LazyProbe have the best performance and outperform the GreedySel baseline by at least 8%. This is because the enumerations can easily find influential billboards when the influence overlap becomes larger.



(a) Influence (b) Efficiency

Fig. 17. Effect of varying trajectory number $|\mathcal{T}|$ on LA.



(a) NYC (b) LA

Fig. 18. Effect of varying λ.

7.4.4 Additional Discussion. We also compared our solution with a meta heuristic algorithm, Simulated Annealing (Annealing), to verify the practical effectiveness. Although Annealing is costly and provides no theoretical bound for our problem, it has been proved to be a very powerful way for most optimization problems and always can find a near optimal solution [22]. Since Annealing is a random search algorithm and its performance is not stable, we run it 50 times for each instance and select the best solution as our baseline. Table 6 reports both the influence value and its relative improvement percentage w.r.t. Annealing for three different choices of budget L. We observe: (1) PartSel and LazyProbe have a very close performance to EnumSel in average. This is because when the overlap between clusters is small, each billboard selected by of PartSel and LazyProbe is less likely to overlap with the billboards in other clusters, and thus the performance of PartSel and LazyProbe would not lose a large accuracy. As discussed later in Section 7.5, EnumSel is very slow to work in practice. (2) PartSel and LazyProbe improve the influence by 6.6% in average as compared to Annealing. (3) TrafficVol which simply uses the traffic volume to select billboards has the worst approximation.

Table 6. Additional Test on NYC

	L=100k		L=200k		L=300k	
Annealing	6,805	0.00%	11,777	0.00%	15,773	0.00%
TrafficVol	5,111	−24.89%	8,520	−27.66%	9,400	−40.40%
GreedySel	6,890	1.25%	12,267	5.56%	16,108	2.12%
EnumSel	7,080	4.04%	13,161	11.75%	16,570	5.05%
PartSel	7,013	3.06%	13,215	12.21%	16,512	4.69%
LazyProbe	7,013	3.06%	13,215	12.21%	16,512	4.69%

7.5 Efficiency Study

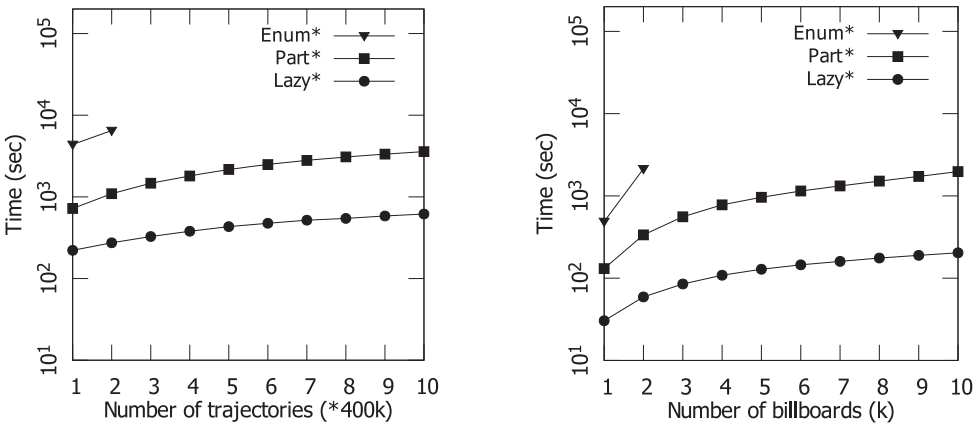
7.5.1 Varying the Budget L. Figures 14(b) and 15(b) present the efficiency result when budget L varies from 100k to 300k on NYC and LA. As EnumSel is too slow to converge in 10^4 seconds (because the complexity of EnumSel is proportional to $|U|^5$), we omit it in the figures. We have three main observations. (1) Lazy* consistently beats Part* by 4–8 times. (2) The gap between Part* and Lazy* is reduced w.r.t. the increase of L. It is because when the more billboards can be afforded, the more local solutions need to be checked as their cost performance is very close. Therefore, the runtime of LazyProbe grows faster than PartSel with L being increased. (3) PartSel and LazyProbe are slower than Lazy* and Part* by nearly one order of magnitude, which indicates that our optimizations works well and would greatly reduce the runtime. (4) TrafficVol is the fastest one with no surprise, because there is no computation involved about the marginal gain evaluations.

7.5.2 Varying the Trajectory Number $\mathcal{T}$. Figures 16(b) and 17(b) show the runtime of all algorithms on NYC and LA datasets. We observe that Part* and Lazy* scale linearly w.r.t. $\mathcal{T}$ which is consistent with our time complexity analysis; moreover, the growth of Part* is faster than Lazy*, e.g., when $|\mathcal{T}|$ varies from $40k$ to $200k$, Lazy* beats Part* 4 and 6.5 times, respectively. Similar to the result in Figures 14(b) and 15(b), PartSel and LazyProbe are much worse than Part* and Lazy* in efficiency.

7.6 Scalability Study

In this experiment, we evaluate the scalability of our methods, Enum*, Part*, and Lazy*, by varying $|\mathcal{T}|$ (from 400k to 4m) and $|U|$ (from 1k to 10k). Since the effectiveness of GreedySel is not satisfying (as evidenced in our effectiveness study), we do not compare the efficiency of GreedySel. The results are shown in Figures 19(a) and 19(b). We can see that Lazy* scales very well and outperforms Part* by 4–8 times. This is because even if the number of billboards is large, Lazy* does not need to compute all local solutions for each cluster with different budgets, while it still can prune a large number of insignificant computations. Since Enum* takes more than 7,200 seconds when the billboard number $|U|$ is larger than 2k, its result is omitted in the Figure for readability reason. It also shows that Enum* has serious issues in efficiency making it impractical in real-world scenarios, while Part* and Lazy* scale well and can meet the efficiency requirement.

Summary. (1) Our methods EnumSel, PartSel, and LazyProbe achieve much higher influence value than existing techniques (GreedySel, TrafficVol, and Annealing). (2) PartSel and LazyProbe achieve similar influence with EnumSel, but EnumSel is too slow and not acceptable in practice while LazyProbe and PartSel are much faster than EnumSel. (3) Compared with the methods without the optimizations proposed in Section 6, Enum*, Part*, and Lazy* are much more efficient, and Lazy* can meet the efficiency requirement on large datasets.



(a) Varying $|\mathcal{T}|$ (b) Varying $|U|$

Fig. 19. Scalability test of our methods on NYC dataset.

8 CONCLUSION

We studied the problem of trajectory-driven influential billboard placement: given a set of billboards U, a database of trajectories $\mathcal{T}$, and a budget L, the goal is to find a set of billboards within L so that the placed ads can influence the largest number of trajectories. We showed that the problem is NP-hard, and first proposed a greedy method with enumeration technique. Then we exploited the locality property of the billboard influence and proposed a partition-based framework PartSel to reduce the computation cost. Moreover, we proposed a lazy probe method LazyProbe to prune billboards with low benefit/cost ratio, which significantly reduces the practical cost of PartSel while achieving the same approximation ratio as PartSel. To further accelerate all the above proposed methods, we introduce an EnumSel early termination and an aggregation index structure to speed up the enumeration and the marginal influence computation. Last we conducted experiments on real datasets to verify the efficiency, effectiveness, and scalability of our methods.

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